Introduction to Determinantal Rings

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Introduction

You are now looking at the notes I prepared for a talk that I gave at BIKES in spring 2024. These notes draw heavily on chapter three of *Determinants, Gröbner Bases and Cohomology* by Bruns, Conca, Raicu, and Varbaro. The figures shown throughout these notes are taken directly from this chapter. If you're looking at this document and find the material interesting, I would definitely recommend giving it a look.

1 Motivation and Basics

Determinants get kind of unwieldy when we're looking at matrices of intdeterminates, so it would be very tricky to study the ideals of matrix minors we'll discuss in this talk if we looked at them explicitly. The following material provides a way of looking at the determinantal rings and ideals that we're interested in while avoiding this potential issue.

1.1 Definitions

We begin by setting up some notation. Let k be a commutative ring – we will actually want it to be a field for most of the contents here, but these *can* be defined over any commutative ring. Our starting point is a matrix of indeterminates

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{m1} & \cdots & X_{mn} \end{pmatrix}.$$

We will usually assume that $m \leq n$. These indeterminates are viewed as living inside the ring $k[X_{11}, \dots, X_{mn}]$, which we will denote by $k[X]$.

An important aspect of X for our purposes will be its t-minors, which are determinants given by selecting t rows and t columns from X. We denote t-minors by

$$\det(X_{a_i b_j}) = [a_1 \dots a_t | b_1 \dots b_t].$$

The **size** of the minor γ, denoted $|\gamma|$, is the number t of rows and columns extracted. In the case where $m \leq n$, the largest minors we can take are m-minors. For such minors we use the notation $[b_1 \dots b_m]$, since all m rows must be used. By convention, the empty minor is set to $1 \in k$.

We denote the ideal of $k[X]$ generated by the t-minors of X by $I_t(X)$, or just I_t if it's clear which X we are discussing. This is the definition of a **determinantal ideal**. A **determinantal ring** is a ring of the form $k[X]/I_t$. Additionally, $\mathcal{M}(X)$ denotes the set of all minors of X and $\mathcal{M}_t(X)$ those of size t.

In general we can replace $k[X]$ with a Noetherian commutative ring R (with some extra hypotheses) and $I_t(X)$ with $I_t(A)$, where A is an $m \times n$ matrix with entries in R. For our purposes, we will focus on the $k[X]$ case.

1.2 Bitableaux

A product of minors Δ is called a **bitableau**. This is due to the association of these products with Young tableaux. A **Young tableau** is a finite collection of cells arranged so that they are aligned on one side with non-increasing numbers of boxes in each successive row. Furthermore, each cell has an entry, usually from a totally ordered set. In this case, we use the integers. These diagrams show up in a number of different places, but perhaps most famously in correspondence with the irreducible representations of symmetric groups.

Now for the association with products of minors in $k[X]$. For

$$\Delta = \delta_1 \cdots \delta_w, \quad \delta_i = [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}]$$

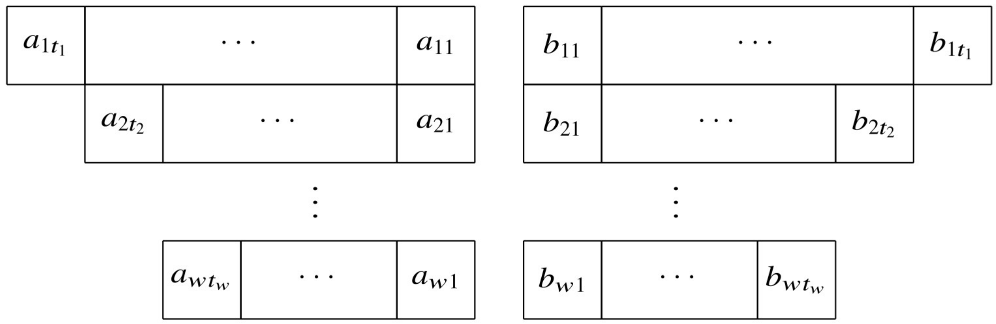

Figure 1: The bitableau associated to the general example on page 2.

we associate the bitableau shown in Figure 1. So, we have a pair of Young Tableaux – one for the row indices used in the corresponding minors, and one for the column indices. The authors of the book say, very honestly, that the figures are made symmetric "for aesthetic reasons."

Recall that $\mathcal{M}(X)$ is the set of all minors of X. We have a partial order on $\mathcal{M}(X)$ given by $[a_1 \dots a_t | b_1 \dots b_t] \preceq [c_1 \dots c_u | d_1 \dots d_u]$ if $t \geq u$ and $a_i \leq c_i$, $b_i \leq d_i$ for all $i \leq u$. A bitableau $\Delta = \delta_1 \cdots \delta_w$ is said to be **standard** if $\delta_1 \preceq \cdots \preceq \delta_w$. This can be interpreted as requiring the entries in each column to be nondecreasing from top to bottom.

2 Standard Bitableaux

2.1 The Straightening Law

We begin this section with the statement of the straightening law, which is probably the central result of this presentation. This theorem comes in four parts, but the main takeaway should be that every element of $k[X]$ has a unique presentation as a k-linear combination of standard bitableaux. In the following statement (and I think for the remainder of the notes), we take k to be a field. This field k can, however, be replaced with a commutative ring and the statement still holds.

Theorem 2.1 (Straightening Law)**.** *(1) The standard bitableaux form a basis of $k[X]$ as a k-vector space.*

(2) (Straightening Relations) If the product $\gamma\delta$ of two minors isn't a standard bitableau, it can be rewritten as

$$\gamma\delta = \sum x_i \epsilon_i \eta_i, \;\; x_i \in k, \;\; x_i \neq 0$$

where $\epsilon_i \eta_i$ is a standard bitableau and $\epsilon_i \prec \gamma$, $\delta \prec \eta_i$.

(3) We can find the standard representation of any bitableau by applying the straightening relations.

(4) If Δ is a bitableau and $\Sigma = \sigma_1 \cdots \sigma_w$ is a standard bitableau appearing in its standard representation, then $\sigma_1 \preceq \delta$ for all factors δ of Δ.

The proof of Theorem 2.1 can be found in the aforementioned textbook. Although Theorem 2.1 (1) states that the standard bitableaux give a basis for all $k[X]$, we will focus mostly on decomposing other bitableaux.

Example 2.2. The following is a nontrivial standard representation for $m = 3$, $n = 6$. Note that we're taking 3-minors, so since $m = 3$ we need only specify the column indices.

$$[1\ 4\ 6][2\ 3\ 5] = [1\ 3\ 5][2\ 4\ 6] - [1\ 2\ 5][3\ 4\ 6] - [1\ 2\ 3][4\ 5\ 6].$$

This computation isn't very easy to check directly, but this is how it decomposes. $\diamondsuit$

In general, it's not really possible to predict what standard bitableaux appear in a given decomposition. However, we can say for sure that the bitableau T_0 given by taking the starting bitableau T and sorting its columns appears with multiplicity one. An example of this sorting is found in Figure 2 below. This can be seen in Example 2.2 as well.

$$T = \begin{array}{|c|c|c|} \hline 2 & 3 & 6 \\ \hline 3 & 4 & 5 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad T_0 = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Figure 2: An example of determining T_0 from a given bitableau T.

In summary, while the process is kind of complicated and not super easy to see, the straightening law gives us a reliable way to decompose any element of $k[X]$ into standard bitableaux.

2.2 Determinantal Ideals

Next, we take a look at some of the applications of standard bitableaux when studying determinantal ideals.

An **ideal** in a partially ordered set $(M, \leq)$ is a subset N so that for all $y \in N$, N also contains all $x \leq y$. Let $\mathcal{N}$ be an ideal in $\mathcal{M}(X)$ (the set of all minors of X), and consider the ideal of $k[X]$ given by $I := \mathcal{N}k[X]$. For $\delta x \in I$, any $\Gamma = \gamma_1 \cdots \gamma_v$ in the standard representation of δx has $\gamma_1 \preceq \delta$, so $\gamma_1 \in \mathcal{N}$.

Proposition 2.3. *In the situation described above, the standard bitableaux $\Gamma = \gamma_1 \cdots \gamma_u$ with $\gamma_1 \in \mathcal{N}$ are a basis of I as a k-vector space.*

Corollary 2.4. *The standard bitableaux $\Gamma = \gamma_1 \cdots \gamma_u$ so that $|\gamma_1| \geq t$ are a basis of I_t as a k-vector space, and the standard bitableaux $\Delta = \delta_1 \cdots \delta_v$ with $|\delta_j| \leq t - 1$ for all j are representatives of a basis for $k[X]/I_t$ as a k-vector space.*

So, we have some nice results about generating sets for these ideals and rings. Next, we take a look at products of determinantal ideals, and in particular their primary decompositions. As it turns out, there's a nice way to approach these using the bitableaux diagrams from earlier. First, some setup. The ideal $\mathfrak{p} := I_t$ is prime in $A := k[X]$. We can get a valuation on the fraction field of A by passing to $P := A_{\mathfrak{p}}$, letting $\mathfrak{q} := \mathfrak{p}P$, and setting

$$v_{\mathfrak{p}}(x) = \max\{i : x \in \mathfrak{q}^i\}$$

where $x \in P \neq 0$, and we set $v_{\mathfrak{p}} = \infty$. It can be shown that $v_{\mathfrak{p}}$ is a discrete valuation on P and can be extended to the fraction field of A.

In the case we have here, i.e. $A = k[X]$, $\mathfrak{p} = I_t$, we denote $v_{\mathfrak{p}}$ by γ_t. It turns out that for a minor δ, this evaluates as

$$\gamma_t(\delta) = \begin{cases} 0, & |\delta| < t \\ |\delta| - t + 1, & |\delta| \geq t \end{cases}.$$

We can extend this to sequences of integers by defining

$$\gamma_t(s_1, \ldots, s_n) = \sum_{i=1}^{n} \max\{s_i - t + 1, 0\}$$

and with this convention, $\gamma_t(|\delta_1|, \ldots, |\delta_n|) = \gamma_t(\delta_1 \cdots \delta_n)$. In terms of Young diagrams, this is simply counting the number of cells in the rightmost t columns. For an example of this, refer to Figure 3 below.

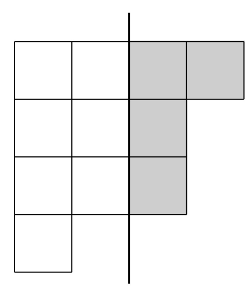

Figure 3: For this bitableau Δ, we have $\gamma_2(\Delta) = 4$.

The ith symbolic power of $\mathfrak{p}$ is

$$\mathfrak{p}^{(i)} = \mathfrak{p}^i P \cap A$$

For $\sigma = (s_1, \ldots, s_n)$, with each s_j being a positive integer, we denote by I^σ the product of ideals $I_{s_1} \cdots I_{s_n}$.

Theorem 2.5. *One has*

$$I_t^{(j)} = \sum_{\sigma=(s_1,\ldots,s_u),\ \gamma_t(\sigma)\geq j} I^\sigma.$$

Furthermore, $I_t^{(j)}$ has a k-basis of all standard bitableaux Σ with $\gamma_t(\Sigma) \geq k$.

Lastly, I'll state the result for primary decomposition of products of determinantal ideals.

Theorem 2.6. *Let $\sigma = (s_1, \ldots, s_n)$ be a nonincreasing sequence of integers and suppose that* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > \min\{r_i, m - r_i, n - r_i\}$. *Then*

$$I^\sigma = \bigcap_{t=1}^{s_1} I_t^{(\gamma_t(\sigma))}.$$

In particular, I^σ is integrally closed.

3 Representation Theory

Here, we briefly discuss a representation used to study determinantal rings and some neat properties of it that result from the bitableaux perspective.

With X an $m \times n$ matrix of indeterminates in $k[X]$ as usual, we define the group $\mathbb{G} := \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$. To each $(A, B) \in \mathbb{G}$, we assign the linear substitution on $k[X]$ given by the map

$$X_{ij} \mapsto \left(AXB^{-1})_{ij}\right)$$

for each entry in X. This gives us a representation $\mathbb{G} \to \mathrm{Aut}(k[X])$.

Using the same notation to describe shapes of bitableaux as in the previous section, we consider shapes $\sigma = (s_1, \ldots, s_n)$ occurring in $k[X]$, i.e. those with $s_1 \leq \min\{m, n\}$. We observe that any t-minor is mapped to a linear combination of t-minors by elements of $\mathbb{G}$, so the determinantal ideals are $\mathbb{G}$-invariant subspaces.

In characteristic 0, we have a nice way of finding subspaces of $k[X]$ that are $\mathbb{G}$-invariant and have a basis of standard bitableaux. A subspace V of $k[X]$ is said to be **defined by shape** if the standard bitableaux contained in V span V, and for any bitableau Δ, whether $\Delta \in V$ or not depends only on the shape $|\Delta|$. When the characteristic of k is 0, V is defined by shape if and only if it is $\mathbb{G}$-invariant and has a basis of standard bitableaux. This does not work in characteristic p.

4 Miscellaneous Curiosities

Here's a quick list of a few connections that I can't say I understand very well, but sound somewhat interesting. I figured they might be worth including in case anybody in the audience knows much about them.

- $K[X]$ and $K[\mathcal{M}_m]$ are examples **algebras with straightening law** on the partially ordered sets $\mathcal{M}(X)$ and $\mathcal{M}_m(X)$, respectively.

- The tools described up to this point are used later in the text to compute Gröbner bases of determinantal ideals. It turns out that the set $\mathcal{M}_m$ of maximal minors of X is a universal Gröbner basis of $I_m(X)$, i.e. a Gröbner basis with respect to every monomial order.

- Determinantal rings are examples of Cohen-Macaulay rings. Wikipedia says that "under mild assumptions," a local ring is Cohen-Macaulay when it is a finitely generated free module over a regular local subring. A regular local ring is a Noetherian local ring with its maximal ideal generated by the same number of elements as its Krull dimension. Alternatively, a ring is Cohen-Macaulay if it is a Cohen-Macaulay module over itself. A **Cohen-Macaulay module** is a module over a commutative noetherian local ring which is finitely generated, nonzero, and its depth equals its Krull dimension. Supposedly, these are "well understood in many ways," and have some properties of a smooth variety.