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On a new invariant for finite groups

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In this article, we define a new invariant for finite groups, called the action-genus. Let G be a finite group. Among all graphs Γ whose automorphism group is isomorphic to G, define the **action-genus** of G to be the minimal genus of a closed connected orientable surface on which Γ can be cellularly embedded. Here, we elucidate some basic properties for the action-genus of a finite group, establish the action-genus of a few infinite families of finite groups, and then conclude with some open questions about the action-genus of finite groups in general.

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1. Introduction

Throughout this article, all groups considered are finite and all graphs considered are finite and simple. The **automorphism group** of a graph Γ, denoted $\operatorname{Aut}\Gamma$, is the set of adjacency preserving permutations of the vertices of Γ. In 1936, König [20] questioned which groups could be realized as the automorphism group of some graph. Three years later, Frucht [8] established that every group may be realized as the automorphism group of some graph. Naturally, this result gave rise to numerous extremal problems in graph theory. Given a group G, there are infinitely many graphs whose automorphism groups are isomorphic to G. Consequently, it is possible to construct graphs with automorphism groups isomorphic to G with arbitrarily large order, size, or genus. It is far more interesting to consider how small a graph can be, and the concept of minimizing graph invariants under certain symmetry restrictions is well-studied.

As an example, there are many results in the study of vertex-minimal graphs with a prescribed automorphism group. For a group G, let $\boldsymbol{\alpha(G)}$ denote the minimum number of vertices among all graphs Γ such that $\operatorname{Aut}\Gamma \cong G$. Babai [3] proved that if G is a group different from the cyclic group of order 3, 4, and 5, then $\alpha(G) \leq 2|G|$. (These three excluded cyclic groups satisfy $\alpha(G) = 3|G|$.) A direct consequence of the results due to Hetzel [18] and Godsil [10, 11] established that Babai's bound can actually be sharpened for most groups. In particular, they proved that $\alpha(G) \leq |G|$ provided G is distinct from each of the following groups: an abelian group of exponent greater than 2; an elementary abelian group of orders 4, 8, or 16; a generalized dicyclic group; and one of ten exceptional groups whose orders are at most 32. In addition to the aforementioned bounds, the exact value of $\alpha(G)$ has been computed for the following infinite families of groups G: abelian groups [1, 25, 31]; hyperoctahedral groups [17]; symmetric groups [27]; alternating groups of degree at least 13 [22]; generalized quaternion groups [13]; dihedral groups [12, 14, 16, 23]; and quasi-abelian and quasi-dihedral groups [21].

The idea of minimizing the size of a graph under certain symmetry restrictions has also been considered. Let $\boldsymbol{e(G,m)}$ denote the minimum number of edges among all graphs Γ with m vertices and $\operatorname{Aut}\Gamma \cong G$; if no such graphs exist, then consider $e(G,m)$ to be undefined. For given group G, the *Minimal-Line Problem* is to determine the value of $e(G,m)$ for each positive integer m. Erdös and Rényi [7] first posed this problem for graphs that have no nontrivial automorphisms. In 1967, Quintas [26] solved the Minimal-Line Problem for the identity group. Of course it is natural to then consider the Minimal-Line Problem for nontrivial groups. The value of $e(G,m)$ is undefined if $m < \alpha(G)$. Moreover, if $m \geq \alpha(G)$ and $m - \alpha(G)$ is small, then the values of $e(G,m)$ can vary greatly. However, for sufficiently large values of m a certain stability is realized. McCarthy and Quintas [24] proved that for each group G, there exists an integer M such that for all $m \geq M$, it is possible to construct a graph on $e(G,m)$ edges with automorphism group isomorphic to G. Nevertheless, the exact value of $e(G,m)$ is only known in a few cases. In particular, $e(G,m)$ has been computed for all integers m provided G is nontrivial and isomorphic to one of the following groups: a symmetric group [27]; the cyclic group of order 3 [9]; a dihedral group of order $2n$, where n is a prime power or twice a prime power [16]; or a hyperoctahedral group [17].

In this article, we are interested in genus-minimal graph embeddings with prescribed automorphism groups. While we have created a new invariant on this topic, the idea of graph embeddings is not new and has received much attention. Recall that the **genus** of a graph Γ, denoted $\gamma(\Gamma)$, is the smallest genus of all the orientable surfaces on which Γ can be embedded. The difficultly of establishing $\gamma(\Gamma)$ is well-known [5], and its complexity was listed as one of the 12 most important open problems in [19]. The *Graph*

Genus Problem asks the following question: given a graph Γ and a positive integer n, does n exceed $\gamma(\Gamma)$? Thomassen [32, 33] established this problem is NP-complete for general graphs and cubic graphs, and that finding the minimum genus of a graph is NP-hard.

Motivated by the aforementioned research on vertex-minimal graphs and edge-minimal graphs with prescribed automorphism groups, we define the action-genus of a group below. Note that the *genus of a group* is similar in name only and an interested reader can see [34] for more information on the genera of groups.

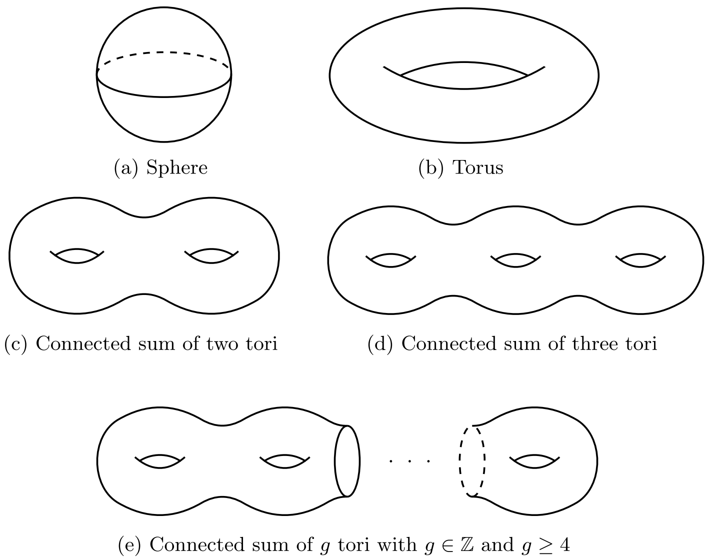
Definition 1.1. Let G be a group. Among all graphs Γ with $\operatorname{Aut}\Gamma \cong G$, define the **action-genus** of G, denoted $\gamma_{\mathfrak{a}}(G)$, to be the minimal genus of a closed connected orientable surface on which Γ can be cellularly embedded.

The definitions of a *closed connected orientable surface* and a *cellular embedding* are stated in Section 2. The action-genus of a group is well-posed because every group may be realized as the automorphism group of some connected graph [8], and every such graph has a cellular embedding in a surface [15]. Thus, every group has an action-genus. This group invariant can be ambitious to compute because, as mentioned above, calculating the genus of a graph is hard and here the genus of *all* graphs with a prescribed automorphism group needs to be considered. Of course, the only exception to this occurs when the action-genus of a group is 0; in this case, establishing one connected planar graph with the prescribed automorphism group is sufficient. As an example, let $n \geq 3$ be an integer and consider the dihedral group of order $2n$. Since the cycle graph of length n can be cellularly embedded in the sphere and has automorphism group isomorphic to D_{2n}, we have that $\gamma_{\mathfrak{a}}(D_{2n}) = 0$.

This article is organized as follows. In Section 2, we develop the background and notation necessary to compute the action-genus of some infinite families of groups; as the action-genus of a group G is a novel group invariant, it is natural to investigate $\gamma_{\mathfrak{a}}(G)$ for some simple cases and we do so in this section. In Section 3, we will establish the action-genus of nontrivial abelian groups. The results of Section 4 establish the action-genus for generalized quaternion groups. Finally, we pose some open questions throughout Section 5, which involve the action-genus of groups in general as well as two extensions of the action-genus of groups.

2. Background and examples

In this article, we are only considering closed connected orientable surfaces. Recall that a **closed surface** is a 2-dimensional compact topological manifold without boundary. Such a surface $\mathcal{S}$ is **connected** provided there is a continuous path on $\mathcal{S}$ between any two points on $\mathcal{S}$. Finally, $\mathcal{S}$ is **orientable** provided an anticlockwise sense of rotation is preserved by traversing any simple closed curve on $\mathcal{S}$ once. It is well-known that every closed connected orientable surface is homeomorphic to a sphere or a connected sum of tori (see, for example, [15]); these homeomorphism classes are depicted in Figure 1. The **genus** of such a surface is the number of tori needed to obtain it through the connected sum operation, where the genus of the sphere is defined to be 0.



(a) Sphere (b) Torus

(c) Connected sum of two tori (d) Connected sum of three tori

(e) Connected sum of g tori with $g \in \mathbb{Z}$ and $g \geq 4$

Figure 1: Homeomorphism classes of closed connected orientable surfaces.

The definition of action-genus of a group requires the associated graphs to be cellularly embedded in the aforementioned surfaces. A graph Γ is **embedded** on a surface $\mathcal{S}$ provided Γ can be represented in $\mathcal{S}$ where the vertices of Γ are distinct points in $\mathcal{S}$ and each edge in Γ is a simple arc connecting its two ends such that no two edges intersect (except possibly at a common end). For example, the complete graph on five vertices, denoted K_5, cannot be embedded on the sphere. However, it can be embedded on the torus; one such embedding is depicted in Figure 2.

Assume that Γ is a graph embedded on a closed connected orientable surface $\mathcal{S}$. In this case, Γ is a topological subspace of $\mathcal{S}$, and thus has a complement. This embedding is **cellular** if the complement of Γ in $\mathcal{S}$ is

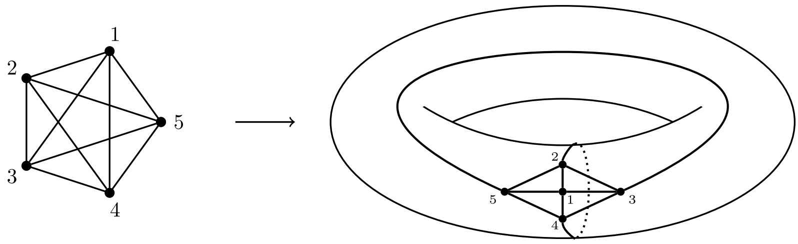


Figure 2: An embedding of K_5 on the torus.

homeomorphic to a disjoint union of open disks. As an example, the cycle graph of length 5 (drawn as a pentagon) can be cellularly embedded on the sphere as its complement is homeomorphic to a disjoint union of two open disks. However, the graph $\Gamma = C_5 \cup K_4$, which has 9 vertices, 11 edges, and is depicted in Figure 3, has no cellular embedding on the sphere. It is

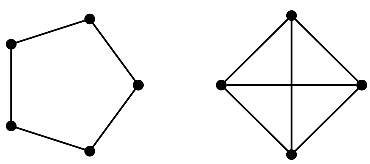
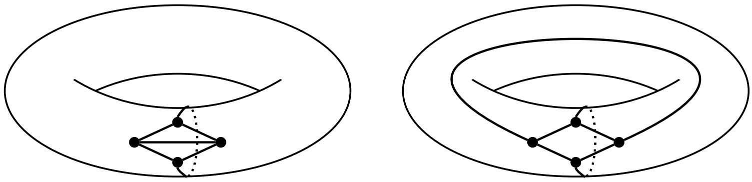


Figure 3: The graph $\Gamma = C_5 \cup K_4$ which as no cellular embedding on the sphere.

not possible to cellularly embed Γ on the sphere because, in any spherical embedding, there is a face of Γ that is not homeomorphic to an open disk. Of note, an embedding of a graph on the sphere is cellular if and only if the graph is connected. However, an embedding of a connected graph on a surface of positive genus may or may not be cellular. For example, Figure 4 depicts two embeddings of the complete graph on four vertices, denoted K_4, on the torus. The embedding in Figure 4(A) is not cellular as one of the faces is homeomorphic to a cylinder; Figure 4(B) depicts a cellular embedding of K_4 on the torus.

With all the necessary terminology for Definition 1.1 in hand, we continue with some more examples of the action-genus of groups.

Example 2.1. For $n \in \mathbb{Z}$ with $n \geq 3$, let S_n denote the symmetric group on n symbols. To compute the value of $\gamma_{\mathfrak{a}}(S_n)$, we must consider all graphs Γ with $\operatorname{Aut}\Gamma \cong S_n$. Certainly, one such graph that comes to mind is K_n, the complete graph on n vertices. However, Ringel and Youngs [30] proved



(a) Non-cellular embedding of K_4 (b) Cellular embedding of K_4

Figure 4: Two embeddings of K_4 on the torus.

that

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

is the minimal genus of a surface on which K_n can be cellularly embedded. Another graph with automorphism group isomorphic to S_n is the complement of K_n — the empty graph on n vertices; but this graph cannot be cellularly embedded on any surface because it is disconnected. Thus, we turn our attention to the star graph on $n+1$ vertices. This graph is depicted in Figure 5, and its automorphism group is isomorphic to S_n. Since

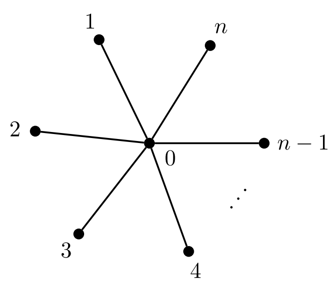


Figure 5: A graph with automorphism group isomorphic to S_n.

the star graph is planar and connected, it can be cellularly embedded on the sphere. Therefore, $\gamma_{\mathfrak{a}}(S_n) = 0$.

Notice that Definition 1.1 does not require the cellular embedding to be closed (a cellular embedding of a graph in a surface is **closed** if each face is bounded by a cycle in the graph). In the next example, we will compute the action-genus of two alternating groups.

Example 2.2. Consider the alternating group on 4 symbols, denoted A_4. Define Γ to be the graph with 36 vertices and 66 edges depicted in Figure 6. A quick computation in SageMath [6] proves that the automorphism group of Γ is generated by the permutations

$$\sigma := (1,10,31)(2,11,32)(3,12,33)(4,13,34)(5,14,35)(6,15,36)(7,16,28)(8,17,29)(9,18,30)(19,22,25)(20,23,26)(21,24,27)$$

and

$$\tau := (1,16)(2,17)(3,18)(4,10)(5,11)(6,12)(7,13)(8,14)(9,15)(19,28)(20,29)(21,30)(22,31)(23,32)(24,33)(25,34)(26,35)(27,36),$$

and that $A_4 \cong \langle \sigma, \tau \rangle$. Since Γ is planar and connected, it can be cellularly

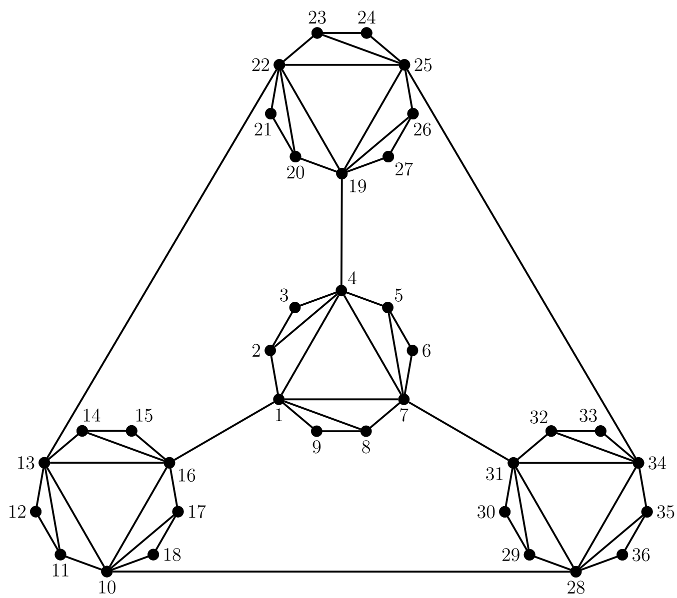


Figure 6: A graph with automorphism group isomorphic to A_4.

embedded on the sphere. Therefore, $\gamma_{\mathfrak{a}}(A_4) = 0$.

Let Γ_9 be the induced subgraph of Γ on the vertices in $\{1, 2, \ldots, 9\}$. The graph Γ was constructed by replacing each vertex of a tetrahedron with Γ_9. In a similar way, if each vertex on a dodecahedron is replaced by Γ_9, the resulting graph will have 180 vertices and 330 edges. Its automorphism group is isomorphic to the alternating group on 5 symbols, denoted A_5, and thus $\gamma_{\mathfrak{a}}(A_5) = 0$.

Part of calculating the action-genus for an infinite family of groups $\{G_n\}_{n=0}^{\infty}$ involves constructing infinitely many graphs Γ_n with $\mathrm{Aut}(\Gamma_n) \cong$

G_n. Establishing the automorphism groups of Γ_n for all $n \in \mathbb{N}$ requires results on both graphs and groups. We review these necessary results now. For the graph Γ, let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set of Γ and edge set of Γ, respectively. An edge between the vertices u and v in Γ is denoted $[u, v]$. In this case, we say that the **ends** of the edge $[u, v]$ are u and v, and that u and v are **adjacent** or **neighbors** in Γ. If $S \subset V(\Gamma)$, then the **induced subgraph** of Γ on S is the graph whose vertex set is S and whose edge set consists of all edges in $E(\Gamma)$ that have both ends in S. The graph $\Gamma - \{v\}$ denotes the induced subgraph on $V(\Gamma)\backslash\{v\}$. In order to establish the automorphism group of a given graph Γ, we will use the Orbit-Stabilizer Theorem, which states the relationship between the order of $\operatorname{Aut}\Gamma$, the size of the orbit of a vertex v in $\operatorname{Aut}\Gamma$, and the order of the stabilizer of v in $\operatorname{Aut}\Gamma$. Specifically, for each $v \in V(\Gamma)$, the **orbit** of v is

$$\mathcal{O}(v) := \{\sigma(v) : \sigma \in \operatorname{Aut}\Gamma\}$$

and the **stabilizer** of v is

$$\operatorname{stab}(v) := \{\sigma \in \operatorname{Aut}\Gamma : \sigma(v) = v\};$$

the Orbit-Stabilizer Theorem states that $|\operatorname{Aut}\Gamma| = |\mathcal{O}(v)| \cdot |\operatorname{stab}(v)|$. Lastly, we require the so-called orbit of an edge in $\operatorname{Aut}\Gamma$. Let $S_{V(\Gamma)}$ denote the symmetric group on the set $V(\Gamma)$. If G is a subgroup of the permutation group $S_{V(\Gamma)}$, then for vertices $u, v \in V(\Gamma)$ the set

$$\mathcal{O}_G\{u, v\} = \{[\sigma(u), \sigma(v)] : \sigma \in G\}$$

defines the **edge orbit** of $[u, v] \in E(\Gamma)$. With these preliminary results in hand, we can now compute the action-genus of nontrivial abelian groups.

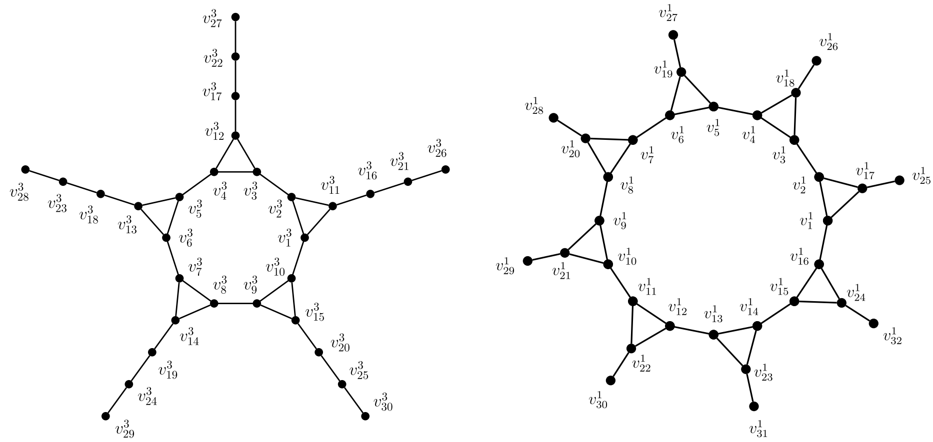
3. Abelian groups

In this section, we will prove that the action-genus of all nontrivial abelian groups G is 0. To this end, we will construct a graph Γ_G with $\operatorname{Aut}(\Gamma_G) \cong G$ that can be cellularly embedded on the sphere. For convenience of the reader, as we construct this graph Γ_G in Definition 3.1 below, a planar embedding is described; since Γ_G is a plane connected graph it has a corresponding cellular embedding on the sphere.

Definition 3.1. Let $n, i \in \mathbb{Z}^+$ with $n \geq 2$. Define the graph $\Gamma(n, i)$ on $3n + in$ vertices with vertex set

$$V\bigl(\Gamma(n, i)\bigr) = \bigl\{v_1^i, v_2^i, \ldots, v_{3n+in}^i\bigr\}$$

and $4n + in$ edges as follows. First, construct a regular $2n$-gon and sequentially label its vertices $v_1^i, v_2^i, \ldots, v_{2n}^i$. Second, for each $j \in \{1, 2, \ldots, n\}$, place the vertex v_{2n+j}^i outside of the $2n$-gon equidistant from the vertices v_{2j-1}^i and v_{2j}^i. Finally, to each vertex v_k^i with $k \in \{2n+1, 2n+2, \ldots, 3n\}$, attach a path of length i that extends radially outward with respect to the center of the $2n$-gon; for each $k \in \{2n+1, 2n+2, \ldots, 3n\}$, sequentially label the vertices in each path $v_{k+n}^i, v_{k+2n}^i, \ldots, v_{k+in}^i$ starting at the vertex closest to vertex v_k^i. The graphs $\Gamma(5, 3)$ and $\Gamma(8, 1)$ are depicted in Figure 7(A) and Figure 7(B), respectively.



(a) Depiction of $\Gamma(5, 3)$ (b) Depiction of $\Gamma(8, 1)$

Figure 7: The graphs $\Gamma(5, 3)$ and $\Gamma(8, 1)$ constructed in Definition 3.1.

Let G be a nontrivial abelian group. By the Fundamental Theorem of Finitely Generated Abelian Groups, there exists integers $a_1, a_2, \ldots, a_m \geq 2$ such that

$$G \cong \mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_m},$$

where $\mathbb{Z}_{a_j}$ denotes the cyclic group of order a_j and $j \in \{1, 2, \ldots, m\}$. Define the graph

$$\hat{\Gamma}_G = \Gamma(a_1, 1) \cup \Gamma(a_2, 2) \cup \cdots \cup \Gamma(a_m, m).$$

Finally, define Γ_G to be the graph with vertex set $V(\Gamma_G) = \{0\} \cup V\big(\hat{\Gamma}_G\big)$ and edge set

$$E(\Gamma_G) = E\big(\hat{\Gamma}_G\big) \cup \big\{[v_{k_j}^j, 0] : \forall j \in \{1, 2, \ldots, m\} \text{ and } k_j \in \{1, 3, \ldots, 2a_j - 1\}\big\}.$$

Note that, by construction, $\Gamma(n,i)$ is a planar graph. Hence, $\hat{\Gamma}_G$ is comprised of m planar components and is also planar. Since the graph Γ_G was then constructed by connecting vertex 0 to a_j vertices in each component $\Gamma(a_j,j)$, the graph Γ_G is connected. Moreover, these additional edges of $E(\Gamma_G)$ in $E(\Gamma_G)\setminus E\big(\hat{\Gamma}_G\big)$ can clearly be included without crossings, yielding a planar embedding of Γ_G.

In the forthcoming lemma, we will prove that the automorphism groups of certain subgraphs of $\Gamma_G - \{0\}$ are cyclic.

Lemma 3.2. *Assume that G is a nontrivial abelian group, and write*

$$G \cong \mathbb{Z}_{a_1}\times\mathbb{Z}_{a_2}\times\cdots\times\mathbb{Z}_{a_m},$$

where $a_1,a_2,\ldots,a_m\geq 2$ are integers. Let Γ_G be the graph constructed in Definition 3.1. For each $j\in\{1,2,\ldots,m\}$, the automorphism group of the induced subgraph of Γ_G on the vertices in $\{0\}\cup\big\{v_1^j,v_2^j,\ldots,v_{3a_j+ia_j}^j\big\}$ is cyclic of order a_j.

Proof. For each $j\in\{1,2\ldots,m\}$, let $\bar{\Gamma}_G(j)$ denote the induced subgraph of Γ_G on the vertices in $\{0\}\cup\big\{v_1^j,v_2^j,\ldots,v_{3a_j+ia_j}^j\big\}$. The permutation

$$\sigma_j := \big(v_1^j,v_3^j,\ldots,v_{2a_j-1}^j\big)\big(v_2^j,v_4^j,\ldots,v_{2a_j}^j\big) \prod_{k=0}^{j}\big(v_{2a_j+1+ka_j}^j,v_{2a_j+2+ka_j}^j,\ldots,v_{3a_j+ka_j}^j\big)$$

which composes of $j+3$ cycles of length a_j, preserves the adjacency relations of Γ_G and thus is an automorphism $\bar{\Gamma}_G(j)$ with order a_j. As a result, $\mathbb{Z}_{a_j}\cong\langle\sigma_j\rangle\leq\operatorname{Aut}\big(\bar{\Gamma}_G(j)\big)$ for each $j\in\{1,2\ldots,m\}$. We will invoke the Orbit-Stabilizer Theorem below to prove that $\operatorname{Aut}\big(\bar{\Gamma}_G(j)\big)=\langle\sigma_j\rangle$.

Note that the vertices in $\bar{\Gamma}_G(j)-\{0\}$ have degree at most 4. Moreover, each vertex in $\big\{v_1^j,v_3^j,\ldots,v_{2a_j-1}^j\big\}$ has degree 4 in Γ_G with at least three degree-3 neighbors. Since vertex 0 is only adjacent to vertices of degree 4, it does not lie in the same orbit as $v_1^j,v_3^j,\ldots,v_{2a_j-1}^j$ under the action of $\operatorname{Aut}\big(\bar{\Gamma}_G(j)\big)$. Thus, the set $\big\{v_1^j,v_3^j,\ldots,v_{2a_j-1}^j\big\}$ actually forms an orbit of $\operatorname{Aut}\big(\bar{\Gamma}_G(j)\big)$ because the action of σ_j on these vertices is transitive. By the Orbit-Stabilizer Theorem,

$$\big|\operatorname{Aut}\big(\bar{\Gamma}_G(j)\big)\big| = \big|\mathcal{O}\big(v_1^j\big)\big|\cdot\big|\operatorname{stab}\big(v_1^j\big)\big| = a_j\cdot\big|\operatorname{stab}\big(v_1^j\big)\big|, \tag{1}$$

and so we examine stab $\left(v_1^j\right)$. Let $\varphi \in \text{stab}\left(v_1^j\right) \leq \text{Aut}\left(\bar{\Gamma}_G(j)\right)$, so that $\varphi(v_1^j) = v_1^j$. The neighbors of v_1^j in Γ_G form an invariant set under φ; in other words,

$$\varphi\left(\{0, v_2^j, v_{2a_j}^j, v_{2a_j+1}^j\}\right) = \{0, v_2^j, v_{2a_j}^j, v_{2a_j+1}^j\}$$

and the induced subgraph of Γ_G on the vertices $0, v_1^j, v_2^j, v_{2a_j}^j$, and $v_{2a_j+1}^j$ is depicted in Figure 8. Notice that $\varphi(0) = 0$ because 0 is the only vertex

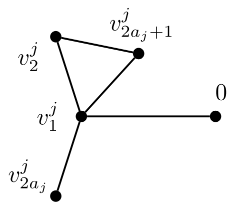


Figure 8: The induced subgraph on the vertex v_1^j and its neighbors.

in $\bar{\Gamma}_G(j)$ all of whose neighbors have degree 4. The vertex v_1^j has only one neighbor, namely $v_{2a_j+1}^j$, that is adjacent to a vertex of degree at most 2, which implies $\varphi(v_{2a_j+1}^j) = v_{2a_j+1}^j$. Consequently, both vertices v_2^j and $v_{2a_j}^j$ are fixed by φ as v_2^j is adjacent to the fixed vertex $v_{2a_j+1}^j$ and $v_{2a_j}^j$ is not. It follows that $\varphi(v_i^j) = v_i^j$ for all $i \in \{1, 2, \ldots, 2a_j\}$ because these vertices compose the only $2a_j$-cycle in $\bar{\Gamma}_G(j)$ whose degree sequence alternates between 4 and 3. In turn, φ then fixes all other vertices in $\bar{\Gamma}_G(j)$. Therefore, φ is the identity element of Aut $\left(\bar{\Gamma}_G(j)\right)$ and $\left|\text{stab}\left(v_1^j\right)\right| = 1$; by Equation (1) we have that $|\,\text{Aut}(\bar{\Gamma}_G(j))| = a_j$. The desired result now follows because $\mathbb{Z}_{a_j} \cong \langle \sigma_j \rangle \leq \text{Aut}\left(\bar{\Gamma}_G(j)\right)$ and $|\mathbb{Z}_{a_j}| = a_j = |\langle \sigma_j \rangle|$. □

We will use this lemma to prove that the graph Γ_G constructed in Definition 3.1 has the proper automorphism group.

Proposition 3.3. *Let G be a nontrivial abelian group. The automorphism group of the graph Γ_G constructed in Definition 3.1 is isomorphic to G.*

Proof. Since G is a nontrivial abelian group, there exists integers

$$a_1, a_2, \ldots, a_m \geq 2$$

such that

$$G \cong \mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_m}$$

by the Fundamental Theorem of Finitely Generated Abelian Groups. Define the permutation

$$\sigma_j := \left(v_1^j, v_3^j, \ldots, v_{2a_j-1}^j\right)\left(v_2^j, v_4^j, \ldots, v_{2a_j}^j\right) \prod_{k=0}^{j}\left(v_{2a_j+1+ka_j}^j, v_{2a_j+2+ka_j}^j, \ldots, v_{3a_j+ka_j}^j\right)$$

for each $j \in \{1, 2, \ldots, m\}$. Let Γ_G be the graph constructed in Definition 3.1; σ_j preserves the adjacency relations of Γ_G and thus is a automorphism of Γ_G with order a_j. As a result, $\langle\sigma_1, \sigma_2, \ldots, \sigma_m\rangle \leq \operatorname{Aut}(\Gamma_G)$, and since $G \cong \langle\sigma_1, \sigma_2, \ldots, \sigma_m\rangle$, it suffices to prove that $|\operatorname{Aut}(\Gamma_G)| = |G|$.

Since vertex 0 in Γ_G is the only vertex whose neighbors all have degree 4, it is fixed under any automorphism of Γ_G. Notice that $\hat{\Gamma}_G$ can be obtained by deleting vertex 0 in Γ_G, and the subgraphs $\Gamma(a_j, j)$, where $j \in \{1, 2, \ldots, m\}$, are the m components of $\hat{\Gamma}_G$. We claim that no two components have the same number of vertices of degree 1 and the same number of vertices of degree 2 — proving that each component is invariant under every automorphism of Γ_G. To this end, recall that the component $\Gamma(a_j, j)$ has a_j vertices of degree 1. If $a_k = a_\ell$ for some distinct $k, \ell \in \{1, 2, \ldots, m\}$, then $\Gamma(a_k, k)$ has $a_k(k-1)$ vertices of degree 2, while $\Gamma(a_\ell, \ell)$ has $a_\ell(\ell-1)$ vertices of degree 2. It follows that $a_k(k-1) \neq a_\ell(\ell-1)$ and each component is invariant under every automorphism of Γ_G. Therefore, these components are the unions of vertex orbits in $\operatorname{Aut}(\Gamma_G)$.

Now consider the induced subgraph of Γ_G on the vertices in $\{0\} \cup \Gamma(a_1, 1)$, denoted by $\bar{\Gamma}_G(1)$. By Lemma 3.2, $\operatorname{Aut}\left(\bar{\Gamma}_G(1)\right)$ is cyclic of order a_1, and so the vertices $v_1^1, v_3^1, \ldots, v_{2a_1-1}^1$ compose an orbit of $\operatorname{Aut}(\Gamma_G)$ because of the transitive action on them by σ_1. By the Orbit-Stabilizer Theorem,

$$|\operatorname{Aut}(\Gamma_G)| = \left|\mathcal{O}\left(v_1^1\right)\right| \cdot \left|\operatorname{stab}\left(v_1^1\right)\right| = a_1 \cdot \left|\operatorname{stab}\left(v_1^1\right)\right|. \tag{2}$$

Thus, we will examine the subgroup $\operatorname{stab}\left(v_1^1\right)$ of $\operatorname{Aut}(\Gamma_G)$. The proof of Lemma 3.2 established that $\operatorname{Aut}\left(\bar{\Gamma}_G(1)\right)$ is generated by the permutation σ_1, and thus any element of $\operatorname{stab}\left(v_1^1\right)$ will fix all of $\Gamma(a_1, 1)$. Consequently, we examine the action of $\operatorname{stab}(v_1^1)$ on the rest of Γ_G. Since σ_j fixes v_1^1 for all $j \in \{2, 3, \ldots, m\}$, we have that $\sigma_j \in \operatorname{stab}(v_1^1)$ provided $j \neq 1$. Moreover, the automorphism group of the induced subgraph of Γ_G on the vertices in $\{0\} \cup \left\{v_1^j, v_2^j, \ldots, v_{3a_j+ia_j}^j\right\}$ is cyclic of order a_j by Lemma 3.2. Because each subgraph $\Gamma(a_j, j)$ is invariant, $\left|\operatorname{stab}\left(v_1^1\right)\right| = a_2 a_3 \cdots a_m$ and Equation (2) implies

$$|\operatorname{Aut}(\Gamma_G)| = a_1 \cdot \left|\operatorname{stab}\left(v_1^1\right)\right| = a_1 a_2 a_3 \cdots a_m = |G|.$$

Since we previously established that

$$G \cong \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \leq \mathrm{Aut}(\Gamma_G),$$

the result now follows. □

With these results in hand, we are now able to prove that the action-genus of every nontrivial abelian group is 0.

Theorem 3.4. *If G is a nontrivial abelian group, then $\gamma_{\mathfrak{a}}(G) = 0$.*

Proof. The graph Γ_G constructed in Definition 3.1 satisfies $\mathrm{Aut}(\Gamma_G) \cong G$ by Proposition 3.3. Since Γ_G is planar and connected by construction, it can be cellularly embedded on the sphere. Therefore, $\gamma_{\mathfrak{a}}(G) = 0$, as desired. □

Next, we will establish an infinite family of groups with positive action-genus.

4. Generalized quaternion groups

In this section, we will establish an infinite family of groups with positive action-genus. For $n \geq 4$ an integer, let Q_{2^n} denote the generalized quaternion group of order 2^n. We will use the following presentation of Q_{2^n}:

$$Q_{2^n} = \left\langle \sigma, \tau : \sigma^{2^{n-1}} = 1 = \tau^4, \tau\sigma\tau^{-1} = \sigma^{-1}, \sigma^{2^{n-2}} = \tau^2 \right\rangle. \tag{3}$$

It is an easy exercise to prove that every element of Q_{2^n} can be expressed as $\sigma^i\tau^j$ for $i \in \{0, 1, \ldots, 2^{n-1} - 1\}$ and $j \in \{0, 1\}$; additionally, $\sigma^{2^{n-2}} = \tau^2$ is the only element of order 2 in Q_{2^n}, and every element in the set $Q_{2^n} \backslash \langle \sigma \rangle$ has order 4.

To prove that $\gamma_{\mathfrak{a}}(Q_{2^n})$ is positive, we proceed as follows. In Definition 4.1, we construct a graph Γ_n for all $n \geq 4$. The results of Proposition 4.2 prove that $\mathrm{Aut}(\Gamma_n) \cong Q_{2^n}$, and we construct a cellular embedding of Γ_n on the torus in Proposition 4.3. An illustrative example for Definition 4.1 and Proposition 4.3 is given in Example 4.4 when $n = 5$. Finally, Theorem 4.5 will prove that $\gamma_{\mathfrak{a}}(Q_{2^n}) = 1$ for all $n \geq 4$.

Definition 4.1. Assume that $n \geq 4$ is an integer. Define the permutations

$$\begin{aligned}\sigma_n \; := \; & (1, 2, \ldots, 2^{n-1})(2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n) \\ & (2^n+1, 2^n+2, \ldots, 3 \cdot 2^{n-1})(3 \cdot 2^{n-1}+1, 3 \cdot 2^{n-1}+2, \ldots, 2^{n+1})\end{aligned}$$

and $\tau_n := \tau_1\tau_2$, where

$$\tau_1 := (1, 2^{n-1}+1, 2^{n-2}+1, 3\cdot 2^{n-2}+1) \prod_{i=2}^{2^{n-2}} (i, 2^n+2-i, 2^{n-2}+i, 3\cdot 2^{n-2}+2-i)$$

and τ_2 is obtained by adding 2^n to each entry in τ_1. Set $G = \langle \sigma_n, \tau_n \rangle$, and notice that G is a subgroup of $S_{2^{n+1}}$. It is easily verified that σ_n and τ_n satisfy the relations of Q_{2^n} given in Equation (3) and thus generate a group isomorphic to Q_{2^n}. Define the graph Γ_n on 2^{n+1} vertices with $V(\Gamma_n) = \{1, 2, \ldots, 2^{n+1}\}$ and where $E(\Gamma_n)$ contains the following four edge orbits:

$$\mathcal{O}_G\{1, 2^{n-1}+1\},\ \mathcal{O}_G\{1, 2^n+1\},\ \mathcal{O}_G\{1, 2^n+2\},\ \text{and}\ \mathcal{O}_G\{1, 3\cdot 2^{n-1}+1\}.$$

Each edge orbit of contains 2^n edges, and thus Γ_n has size $4 \cdot 2^n = 2^{n+2}$.

The graph Γ_5 is constructed in Example 4.4 and depicted in Figure 12. In the forthcoming proposition, we will establish the automorphism group of this graph Γ_n for all $n \geq 4$.

Proposition 4.2. *For $n \geq 4$, the graph Γ_n constructed in Definition 4.1 has automorphism group isomorphic to Q_{2^n}.*

Proof. Let Γ_n, σ_n, and τ_n be as given in Definition 4.1, and recall that $\langle \sigma_n, \tau_n \rangle \cong Q_{2^n}$. Since the edges in Γ_n are the images of the edges

$$[1, 2^{n-1}+1],\ [1, 2^n+1],\ [1, 2^n+2],\ \text{and}\ [1, 3\cdot 2^{n-1}+1]$$

under the elements of $\langle \sigma_n, \tau_n \rangle$, the permutations σ_n and τ_n will preserve all adjacency relations of Γ_n. Therefore, $Q_{2^n} \cong \langle \sigma_n, \tau_n \rangle \leq \mathrm{Aut}(\Gamma_n)$, and it suffices to prove that $|\mathrm{Aut}(\Gamma_n)| = 2^n$. To this end, partition $V(\Gamma_n)$ into the following two sets:

$$V_1 := \{1, 2, \ldots, 2^n\} \quad \text{and} \quad V_2 := \{2^n+1, 2^n+2, \ldots, 2^{n+1}\}.$$

The vertices in V_1 and V_2 have degree 5 and 3, respectively. Consequently, V_1 and V_2 are invariant sets under the action of $\mathrm{Aut}(\Gamma_n)$. Since $\langle \sigma_n, \tau_n \rangle \cong Q_{2^n}$ acts transitively on the sets V_1 and V_2, the orbits of $\mathrm{Aut}(\Gamma_n)$ are V_1 and V_2, and

$$|\mathrm{Aut}(\Gamma_n)| = |\mathcal{O}(1)| \cdot |\mathrm{stab}(1)| = 2^n \cdot |\mathrm{stab}(1)| \tag{4}$$

by the Orbit-Stabilizer Theorem.

To prove that $|\mathrm{stab}(1)| = 1$, let $\varphi \in \mathrm{stab}(1) \leq \mathrm{Aut}(\Gamma_n)$. Since 1 is fixed by φ, its neighbors compose an invariant set under φ; in other words, the set

$$\{2^{n-1}+1, 3\cdot 2^{n-2}+1, 2^n+1, 2^n+2, 3\cdot 2^{n-1}+1\}$$

is fixed setwise by φ. Consider the induced subgraph of Γ_n on vertex 1 and its neighbors, denoted by $\bar{\Gamma}_n(1)$, which is depicted in Figure 9. Since 2^n+2

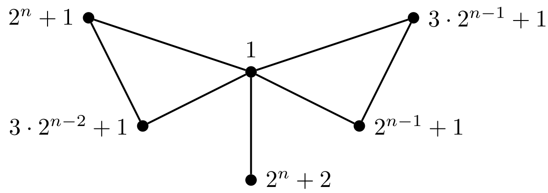


Figure 9: The induced subgraph of Γ_n on vertex 1 and its neighbors.

is the only degree-1 vertex in $\bar{\Gamma}_n(1)$, $\varphi(2^n+2) = 2^n+2$. It follows that the sets of vertices $\{2^{n-1}+1, 3\cdot 2^{n-2}+1\}$ and $\{2^n+1, 3\cdot 2^{n-1}+1\}$ are invariant under φ because the first set is contained in V_1 and the latter set is contained in V_2. To establish that these sets are actually fixed pointwise by φ, consider the induced subgraph of Γ_n on the vertices at most distance 2 from vertex 1; this graph is depicted in Figure 10. Since vertex $7\cdot 2^{n-2}$ is

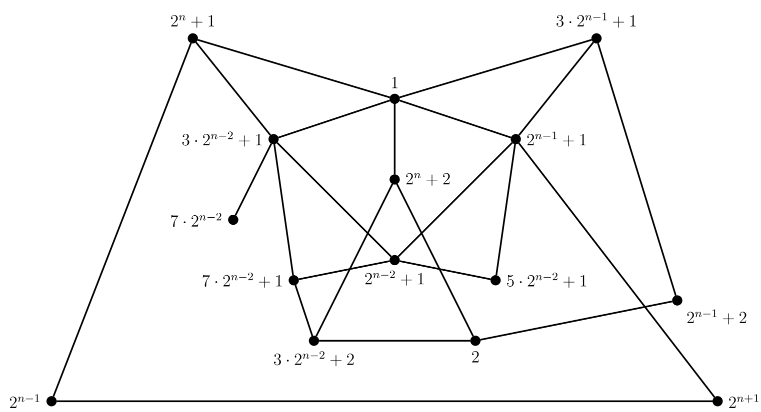


Figure 10: The induced subgraph of Γ_n on the vertices at most distance 2 from vertex 1.

the only vertex with degree 1 in this subgraph, it is fixed by φ. It follows that its only neighbor satisfies $\varphi(3\cdot 2^{n-2}+1) = 3\cdot 2^{n-2}+1$ and thus all the neighbors of 1 are fixed under φ. Lastly, note that $\varphi(2) = 2$ because in this subgraph it is the only neighbor of the fixed vertex 2^n+2 with a neighbor of degree 2. In summary, $\varphi(1) = 1$ implies that vertices 2, $2^{n-1}+1$, 2^n+1, and $3 \cdot 2^{n-1}+1$ are fixed by φ.

Repeating the argument above with vertex 1 replaced by vertex 2, which is possible as these vertices lie in the same orbit under $\mathrm{Aut}(\Gamma_n)$ and $\varphi(2) = 2$, yields that vertices 3, $2^{n-1}+2$, 2^n+2, and $3 \cdot 2^{n-1}+2$ are fixed by φ. Continuing this process by replacing i with $i+1$ for $i \in \{3, 4, \dots, 2^{n-1}\}$, proves that all vertices of Γ_n are fixed by $\varphi \in \mathrm{stab}(1)$. Therefore, φ is the trivial automorphism of $\mathrm{Aut}(\Gamma_n)$ and $|\mathrm{stab}(1)| = 1$. Equation (4) then implies that $|\mathrm{Aut}(\Gamma_n)| = 2^n$. Because we previously established that $Q_{2^n} \cong \langle \sigma_n, \tau_n \rangle \leq \mathrm{Aut}(\Gamma_n)$, it follows that $\mathrm{Aut}(\Gamma_n) \cong Q_{2^n}$ for all $n \geq 4$. □

The graph Γ_n constructed in Definition 4.1, which satisfies $\mathrm{Aut}(\Gamma_n) \cong Q_{2^n}$ by Proposition 4.3, will be used to establish that $\gamma_{\mathfrak{a}}(Q_{2^n}) \leq 1$. In the proposition that follows, we will cellularly embedded Γ_n on the torus.

Proposition 4.3. *The graph Γ_n constructed in Definition 4.1 can be cellularly embedded on the torus for all $n \geq 4$.*

Proof. Since graph embeddings on the torus can be difficult to visualize, in this proof we will utilize the planar representation of the torus (which depicts the torus through identifying the opposite sides of a rectangle). To describe our embedding of Γ_n for all $n \geq 4$, we first discuss the placement of the vertices of Γ_n in the rectangle and then discuss how to draw the edges of Γ_n.

Partition the vertices of Γ_n into eight rows of size 2^{n-2}, where each row contains the following vertices.

Row 1: $1, 2, \dots, 2^{n-2}$
Row 2: $2^n+1, 2^n+2, \dots, 5 \cdot 2^{n-2}$
Row 3: $3 \cdot 2^{n-1}+1, 3 \cdot 2^{n-1}+2, \dots, 7 \cdot 2^{n-2}$
Row 4: $2^{n-1}+1, 2^{n-1}+2, \dots, 3 \cdot 2^{n-2}$
Row 5: $2^{n-2}+1, 2^{n-2}+2, \dots, 2^{n-1}$
Row 6: $5 \cdot 2^{n-2}+1, 5 \cdot 2^{n-2}+2, \dots, 3 \cdot 2^{n-1}$
Row 7: $7 \cdot 2^{n-2}+1, 7 \cdot 2^{n-2}+2, \dots, 2^{n+1}$
Row 8: $3 \cdot 2^{n-2}+1, 3 \cdot 2^{n-2}+2, \dots, 2^n$

In the rectangle representing the torus, fix a positive distance d and draw consecutively labelled vertices within each row at distance d apart such that:

1. Row a is positioned above Row $a+1$ for all $a \in \{1, 2, \dots, 7\}$;

2. Vertices $5\cdot 2^{n-2}+1$ and $3\cdot 2^{n-2}+1$ are aligned vertically and lie farthest to the left;
3. Vertices 2^n+1, $2^{n-1}+1$, $2^{n-2}+1$, and $7\cdot 2^{n-2}+1$ lie on the perpendicular bisector of the vertices $5\cdot 2^{n-2}+1$ and $5\cdot 2^{n-2}+2$; and
4. Vertices 1 and $3\cdot 2^{n-1}+1$ align vertically with vertex $5\cdot 2^{n-2}+2$.

In this case, the vertices in Rows 1 and 3 are aligned vertically, the vertices in Rows 2, 4, 5, and 7 are aligned vertically, and the vertices in Rows 6 and 8 are aligned vertically.

To describe the placement of the edges in Γ_n for this embedding, we will partition $E(\Gamma_n)$ into two sets. Define

$$E_1 := \left\{\sigma^i\tau[1,2^{n-1}+1], \sigma^i\tau[1,3\cdot 2^{n-1}+1] : i\in\{2^{n-2},\dots,2^{n-1}-1\}\right\} \cup \left\{\omega[1,2^n+2] : \omega\in\left\{\sigma^{2^{n-2}-1}, \sigma^{2^{n-2}}\tau, \sigma^{2^{n-1}-1}, \tau\right\}\right\} \subset E(\Gamma_n)$$

and $E_2 := E(\Gamma_n)\backslash E_1$. The set E_1 will contain the edges of Γ_n that cross a boundary of the rectangle that represents the torus in our embedding, while E_2 contains all other edges of Γ_n, which will cross no boundary of the rectangle. We note that E_1 contains $2^{n-1}+4$ of the 2^{n+2} edges of Γ_n. Draw each edge $[u,v]\in E_2$ to lie along the line segment that represents the shortest distance between vertices u and v. This will result in three rows of $2^{n-2}-1$ cycles of length 6 each with an additional cord, which are depicted in Figure 11, the path on vertices 1 and 2^n+1, and the three triangles

$$(1,2^{n-1}+1,3\cdot 2^{n-1}+1),\quad (2^{n-1}+1,5\cdot 2^{n-2}+1,2^{n-2}+1),$$

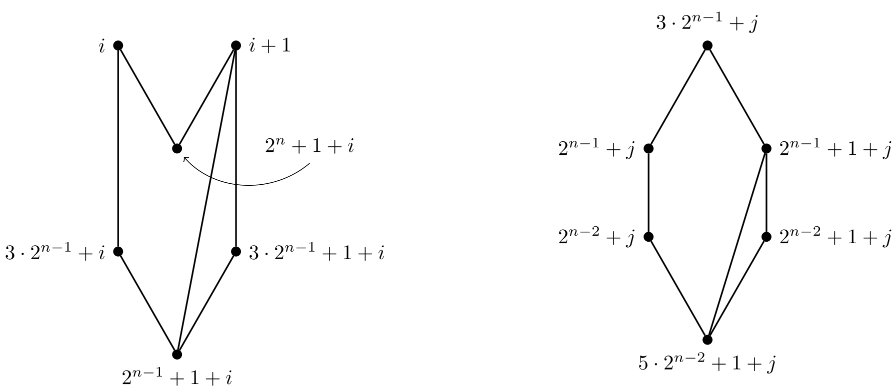


Figure 11: In the proof of Proposition 4.3: The $3(2^{n-2}-1)$ subgraphs of Γ_n created by the inclusion of the edges in E_2, where $i\in\{1,2,\dots,2^{n-2}-1\}\cup\{2^{n-2}+1,2^{n-2}+2,\dots,2^{n-1}-1\}$ and $j\in\{1,2,\dots,2^{n-2}-1\}$.

and

$$(2^{n-2}+1, 3\cdot 2^{n-2}+1, 7\cdot 2^{n-2}+1).$$

Next, the 2^{n-1} edges of E_1 in

$$\left\{\sigma^i\tau[1, 2^{n-1}+1], \sigma^i\tau[1, 3\cdot 2^{n-1}+1] : i\in\{2^{n-2}, 2^{n-2}+1, \dots, 2^{n-1}-1\}\right\}$$

can be drawn without crossings over the identified top and bottom edges of the rectangle that represents the torus. Reading from left to right, note that these edges will cross this boundary of the rectangle in the following order:

$$\sigma^{2^{n-2}}\tau[1, 3\cdot 2^{n-1}+1],\ \sigma^{2^{n-2}}\tau[1, 2^{n-1}+1],\ \sigma^{2^{n-2}+1}\tau[1, 3\cdot 2^{n-1}+1],$$
$$\sigma^{2^{n-2}+1}\tau[1, 2^{n-1}+1],\ \dots,\ \sigma^{2^{n-1}-1}\tau[1, 3\cdot 2^{n-1}+1], \sigma^{2^{n-1}-1}\tau[1, 2^{n-1}+1].$$

Finally, we consider the four remaining edges of E_1, namely those in

$$\left\{\omega[1, 2^n+2] : \omega\in\left\{\sigma^{2^{n-2}-1}, \sigma^{2^{n-2}}\tau, \sigma^{2^{n-1}-1}, \tau\right\}\right\}.$$

The edges $\sigma^{2^{n-1}-1}[1, 2^n+2]$ and $\tau[1, 2^n+2]$ can be included in our embedding of Γ_n without crossings over the identified left and right sides of the rectangle. Starting at vertex $\sigma^{2^{n-2}-1}(1) = 2^{n-2}$, draw the edge $\sigma^{2^{n-2}-1}[1, 2^n+2]$ over the top of the rectangle, then over the right side of the rectangle ending at vertex $\sigma^{2^{n-2}-1}(2^n+2) = 5\cdot 2^{n-2}+1$. Lastly, starting at the vertex $\sigma^{2^{n-2}}\tau(1) = 3\cdot 2^{n-2}+1$, without any edge crossings draw the edge $\sigma^{2^{n-2}}\tau[1, 2^n+2]$ over the bottom of the rectangle, then over the left side of the rectangle ending at vertex $\sigma^{2^{n-2}}\tau(2^n+2) = 7\cdot 2^{n-2}$. Since all faces in this embedding of Γ_n are polygons, it is a cellular embedding for all $n\geq 4$, as desired. □

Before proving the main result of this section, we give an illustrative example of Definition 4.1 and Proposition 4.3 when $n = 5$.

Example 4.4. Assume that $n = 5$. The permutations σ_5 and τ_5 stated in Definition 4.1 are as follows.

$$\sigma_5 := (1, 2, \dots, 16)(17, 18, \dots, 32)(33, 34, \dots, 48)(49, 50, \dots, 64)$$

$$\begin{aligned}\tau_1 &:= (1, 17, 9, 25)\prod_{i=2}^{8}(i, 34-i, 8+i, 26-i)\\ &= (1, 17, 9, 25)(2, 32, 10, 24)(3, 31, 11, 23)(4, 30, 12, 22)\\ &\quad (5, 29, 13, 21)(6, 28, 14, 20)(7, 27, 15, 19)(8, 26, 16, 18)\end{aligned}$$

$$\begin{aligned}\tau_2 &:= (33,49,41,57)(34,64,42,56)(35,63,43,55)(36,62,44,54)\\ &\quad (37,61,45,53)(38,60,46,52)(39,59,47,51)(40,58,48,50)\\ \tau_5 &:= \tau_1\tau_2\end{aligned}$$

The graph Γ_5 has 64 vertices with vertex set $V(\Gamma_5) = \{1, 2 \ldots, 64\}$. Since $G = \langle \sigma_5, \tau_5\rangle \cong Q_{32}$ is a subgroup of S_{64}, the 128 edges of Γ_5 are

$$E(\Gamma_5) = \mathcal{O}_G\{1, 17\} \cup \mathcal{O}_G\{1, 33\} \cup \mathcal{O}_G\{1, 34\} \cup \mathcal{O}_G\{1, 49\}.$$

As an example, the edge orbit

$$\mathcal{O}_G\{1, 17\} = \{[\omega(1), \omega(17)] : \omega \in G = \langle \sigma_5, \tau_5\rangle\}$$

contains 32 edges; these edges are obtained by applying each of the elements in G, namely $\sigma_5^0 = 1, \sigma_5, \sigma_5^2, \ldots, \sigma_5^{15}, \tau_5, \sigma_5\tau_5, \sigma_5^2\tau_5, \ldots, \sigma_5^{15}\tau_5$, to the edge $[1, 17]$ to obtain the edges

$$\begin{aligned}&[1,17],\ [2,18],\ [3,19],\ [4,20],\ [5,21],\ [6,22],\ [7,23],\ [8,24],\\ &[9,25],\ [10,26],\ [11,27],\ [12,28],\ [13,29],\ [14,30],\ [15,31],\ [16,32],\\ &[9,17],\ [10,18],\ [11,19],\ [12,20],\ [13,21],\ [14,22],\ [15,23],\ [16,24],\\ &[1,25],\ [2,26],\ [3,27],\ [4,28],\ [5,29],\ [6,30],\ [7,31],\ [8,32],\end{aligned}$$

respectively. The graph Γ_5 is depicted in Figure 12; its automorphism group is isomorphic to Q_{32} by Proposition 4.2.

Next, we will use the process described in the proof of Proposition 4.3 to embed Γ_5 in the torus. Partition the vertices of Γ_5 into eight rows of size 8, where each row contains the following vertices.

Row 1:	1, 2, 3, 4, 5, 6, 7, 8	Row 5:	9, 10, 11, 12, 13, 14, 15, 16
Row 2:	33, 34, 35, 36, 37, 38, 39, 40	Row 6:	41, 42, 43, 44, 45, 46, 47, 48
Row 3:	49, 50, 51, 52, 53, 54, 55, 56	Row 7:	57, 58, 59, 60, 61, 62, 63, 64
Row 4:	17, 18, 19, 20, 21, 22, 23, 24	Row 8:	25, 26, 27, 28, 29, 30, 31, 32

In the rectangle representing the torus, fix a positive distance d and draw the consecutively labelled vertices within each row at distance d apart with:

1. Vertices 41 and 25 aligned vertically and positioned farthest to the left;
2. Vertices 33, 17, 9, and 57 positioned on the perpendicular bisector of vertices 41 and 42; and
3. Vertices 1 and 49 aligned vertically with vertex 42.

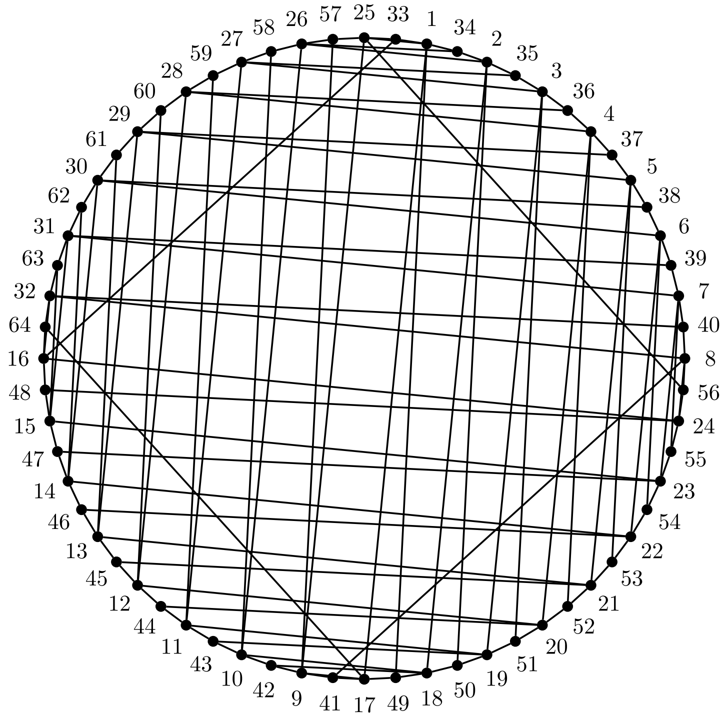


Figure 12: The graph Γ_5, which was constructed in Definition 4.1.

Now, partition the edges of Γ_5 into the following two sets:

$$\begin{aligned} E_1 \ &:= \left\{\sigma^i\tau[1,17], \sigma^i\tau[1,49] : i \in \{8,9,\dots,15\}\right\} \\ &\cup \left\{\omega[1,34] : \omega \in \left\{\sigma^7, \sigma^8\tau, \sigma^{15}, \tau\right\}\right\} \end{aligned}$$

and $E_2 := E(\Gamma_5)\backslash E_1$. Recall that the set E_1 contains the edges of Γ_5 that will cross a boundary of the rectangle that represents the torus, and E_2 contains all other edges of Γ_5, which will cross no boundary of the rectangle. Draw each edge $[u, v] \in E_2$ to lie along the line segment that represents the shortest distance between vertices u and v. This will result in three rows of seven 6-cycles each with an additional cord, the edge $[1, 33]$, and the three triangles $(1, 17, 49)$, $(17, 41, 9)$, $(9, 25, 57)$. The 16 edges of E_1 in

$$\left\{\sigma^i\tau[1,17], \sigma^i\tau[1,49] : i \in \{8,9,\dots,15\}\right\}$$

can now be drawn without crossings over the identified top and bottom boundaries of the rectangle that represents the torus; reading from left to right, these edges will cross the boundary of the rectangle in the following order:

$$[25,33],\ [1,25],\ [26,34],\ [2,26],\ [27,35],\ [3,27],\ [28,36],\ [4,28],$$
$$[29,37],\ [5,29],\ [30,38],\ [6,30],\ [31,39],\ [7,31],\ [32,40],\ [8,32].$$

Finally, we consider the four remaining edges of E_1:

$$\sigma^7[1,34]=[8,41],\ \sigma^8\tau[1,34]=[25,56],\ \sigma^{15}[1,34]=[16,33]$$

and

$$\tau[1,34]=[17,64].$$

Include the edges $[16,33]$ and $[17,64]$ in our embedding of Γ_5 without crossings over the identified left and right sides of the rectangle. Starting at vertex 8, draw the edge $[8,41]$ over the top of the rectangle, then over the right side of the rectangle ending at vertex 41. Lastly, starting at the vertex 25, draw the edge $[25,56]$ over the bottom of the rectangle, then over the left side of the rectangle ending at vertex 56 without any edge crossings. This cellular embedding of Γ_5 on the rectangular representation of the torus is depicted in Figure 13.

We will conclude this section with a proof that the action-genus of every generalized quaternion group Q_{2^n} with $n \geq 4$ is 1.

Theorem 4.5. *If $n \geq 4$ is an integer, then $\gamma_{\mathfrak{a}}(Q_{2^n}) = 1$.*

Proof. Let Γ_n be the graph constructed in Definition 4.1. By Proposition 4.2, we have that $\text{Aut}(\Gamma_n) \cong Q_{2^n}$ for all $n \geq 4$. It follows that $\gamma_{\mathfrak{a}}(Q_{2^n}) \leq 1$ because Γ_n has a cellular embedded on the torus by Proposition 4.3. Babai [2] proved that no graph whose automorphism group is isomorphic to a generalized quaternion group is planar. Consequently, $\gamma_{\mathfrak{a}}(Q_{2^n}) > 0$ and thus $\gamma_{\mathfrak{a}}(Q_{2^n}) = 1$, as desired. □

5. Discussion and open questions

In this section, we pose four open questions that involve the action-genus of groups. Our main focus in this article has been to compute the action-genus of a given infinite family of groups; these results produced infinite collections of groups with small action-genus. It is natural to ask the following question.

Open Question 1. Given a positive integer n, does there exist a group G with $\gamma_{\mathfrak{a}}(G) = n$?

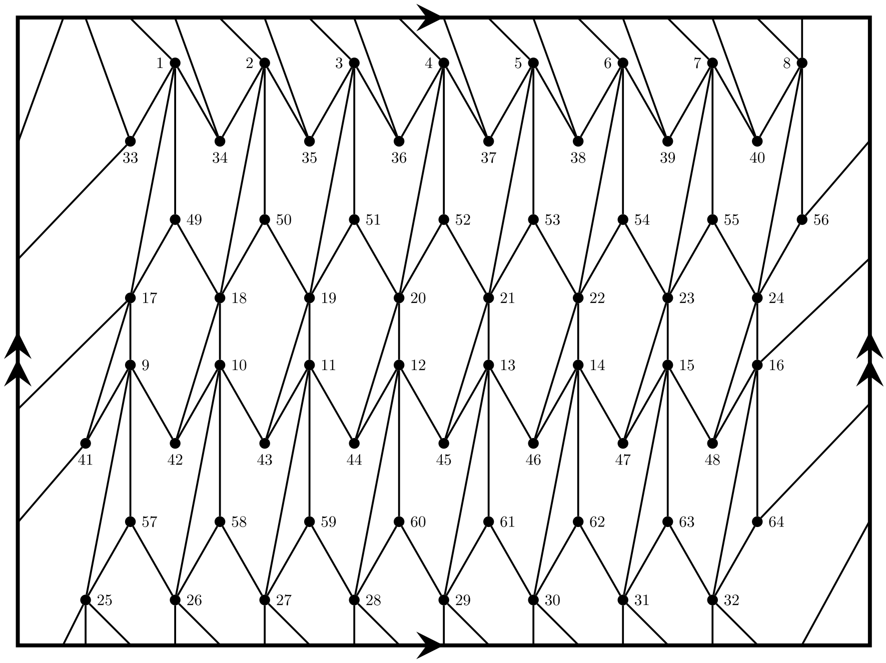


Figure 13: The graph Γ_5, which was constructed in Definition 4.1, embedded on the torus; here, we use the planar representation of the torus where the opposite sides of a rectangle have been identified.

In the past, there was great interest in computing the genera of infinite families of graphs. For example, Beineke and Harary [4] established the genus of the hypercube graph, and Rignel [29, 28] calculated the genus of the complete bipartite graph. Ringel and Youngs [30] computed the genus of the complete graph, which solved the Heawood Map-Coloring Problem. There has also been some research on computing the genera of tripartite graphs; however only partial results have been established (see [34] for more information). For each of these aforementioned families of graphs, the genera grows without bound. We wonder if this is a property that the action-genus of groups can also exhibit.

Open Question 2. Does there exist an infinite family of groups $\{G_n\}_{n=0}^{\infty}$ such that $\gamma_{\mathfrak{a}}(G_n)$ is unbounded?

The remaining two open questions involve extensions of the action-genus of groups and are based on prior work completed in topological graph theory. If Γ is a connected graph, the **maximum genus** of Γ is the largest genus of all the orientable surfaces on which Γ can be cellularly embedded. Motivated by the work on this invariant, we make the following definition for groups G. Among all graphs Γ with $\operatorname{Aut}\Gamma \cong G$, define the **maximum action-genus** of G, denoted $\gamma_{\mathfrak{a}}^{M}(G)$, to be the maximal genus of a closed connected orientable surface on which Γ can be cellularly embedded. In Example 2.1, we proved that $\gamma_{\mathfrak{a}}(S_4) = 0$; the cellular embedding of K_4 in the torus depicted in Figure 4(B) shows that $\gamma_{\mathfrak{a}}^{M}(S_4) \geq 1$. Consequently, we ask the following question.

Open Question 3. Let G be a group. What is the value of $\gamma_{\mathfrak{a}}^{M}(G)$? How does it compare to the value of $\gamma_{\mathfrak{a}}(G)$?

There are two types of closed surfaces, and in this article we have only considered one type — orientable surfaces. The second type of closed surfaces are called non-orientable surfaces, all of which have been classified. Every closed connected non-orientable surface can be obtained by cutting holes in a sphere and closing off each hole using a Möbius band (see, for example, [15]). The **crosscap number** of a closed connected non-orientable surface is the number of Möbius bands used to obtain its homeomorphism type. For $k \in \mathbb{N}$, let N_k denote a closed connected non-orientable surface with crosscap number k. The **crosscap number** (or **non-orientable genus**) of a graph Γ is the minimal k such that Γ can be embedded in N_k. Influenced by these definitions, we make the following definition for groups G. Among all graphs Γ with $\operatorname{Aut}\Gamma \cong G$, define the **non-orientable action-genus** of G, denoted $\tilde{\gamma}_{\mathfrak{a}}(G)$, to be the minimal non-orientable genus of a closed connected non-orientable surface on which Γ can be embedded. We conclude this article with the following question.

Open Question 4. Let G be a group. What is the value of $\tilde{\gamma}_{\mathfrak{a}}(G)$? How does it compare to the value of $\gamma_{\mathfrak{a}}(G)$?

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