CENTRALIZER ALGEBRAS AND THEIR APPLICATIONS

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Abstract. The centralizer algebra of a representation of a finite group G on a complex vector space V is the algebra of all endomorphisms of V that commute with the endomorphisms obtained from the representation. By studying centralizer algebras, we can obtain information about the irreducible representations of G. In general it is not easy to extract meaningful information from the centralizer algebra. However, in the case of a permutation representation there are convenient combinatorial descriptions of the centralizer algebra's structure. In this thesis, I describe some known methods for investigating these algebras. I also include several functions for performing relevant computations in GAP and examples of their use.

Contents

1. Introduction

Representations are one of the most useful tools in the modern study of groups. The ability of representation theory to frame questions about groups in terms of linear algebra is widely applicable.

Using the equivalence of complex representations of a group G and $\mathbb{C}G$-modules, one might consider the endomorphisms of a $\mathbb{C}G$-module V, called its centralizer algebra. As it is possible to discern information about a ring from studying its modules and their endomorphisms, it seems reasonable to expect that some understanding about G or its representations could be discerned by studying these. In practice though, this is not generally a fruitful endeavor. It is difficult to obtain any knowledge from studying the centralizer algebra of a $\mathbb{C}G$-module.

One of the most basic types of representation to study is a permutation representation. A permutation representation arises from the action of a group on a set, and as such is tied to the notion of a group as a realization of symmetries. Of particular interest to us is the centralizer algebra of the corresponding $\mathbb{C}G$-module. Even more specifically, we focus on the situation where this centralizer algebra is commutative.

In this special case, we find that it is possible to describe the structure of the centralizer algebra and make use of it to obtain information about the representations of G which are the direct summands in the permutation representation. Questions such as finding a basis for the centralizer algebra reduce to straightforward combinatorial formulas. The methods used to do so lend themselves to a computer-based implementation, which makes them appealing. Our primary objective is to detail how these methods are used to determine the degrees of irreducible representations of the group G.

Centralizer algebras have many other uses beyond the scope of this thesis. There are many examples of theorems where properties of the endomorphisms of an A-module are used to obtain information about the module itself. See, for example, the Krull-Schmidt theorem [6, p. 65]. Centralizer algebras are also applicable to questions in computational group theory. For example, Nickerson implemented the Split-P algorithm for decomposing permutation modules and made use of it to classify many group representations over fields of characteristic zero [7]. Lastly Hecke algebras, which can be thought of as a more general form of centralizer algebras, also appear in representation theory [6].

2. Background and Examples

The objects that lie at the core of the mathematics outlined in this thesis are groups. Groups appear naturally in geometry. For example, the symmetries of a regular polygon can be realized as a group [1].

The study of groups can be framed as a generalization of the idea of symmetry. Consequently, groups are useful in the natural sciences as well as in strictly mathematical settings. The Poincaré group consists of transformations of Minkowski spacetime and describes the symmetries inherent in the theory of special relativity [8]. Groups also manifest in chemistry [2] and crystallography [4].

Definition 2.1. A **group** is a set G together with an associative binary operation such that G has an identity element e and each element $g \in G$ has an inverse, denoted g^{-1}. A group is **abelian** if this operation is also commutative. A **subgroup** of a group G is a subset H of G which is also a group under the same operation. Given groups G and H, a **group homomorphism** from G to H is a map $\varphi : G \to H$ such that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for all $g_1, g_2 \in G$. If there is a bijective group homomorphism from G to H then we say G is **isomorphic** to H.

Definition/Example 2.2. As an example of a group, consider the set of all bijections from a set X to itself together with the operation of function composition. This operation is associative. There is an identity element, namely the identity function. Lastly, each bijection is invertible. This group is referred to as the **symmetric group** of X and denoted by S_X. The symmetric group of $\{1, \ldots, n\}$ is denoted S_n. $\diamondsuit$

We also consider structures that involve more than one operation. For example, in the complex numbers we can add, multiply, and invert both of these operations (with the exception of zero, which lacks a multiplicative inverse). Rings can be thought of as a generalization of this sort of structure, having a few less requirements.

Definition 2.3. A **ring** is a set K together with two operations, $\cdot$ and $+$, such that K is an abelian group under $+$, the operation $\cdot$ is associative and has an identity, and $\cdot$ has an identity and is satisfies the following equations for all $a, b, c \in K$.

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a)$$

If the operation $\cdot$ is also commutative, we call K a **commutative ring**. Elements in K that have an inverse element with respect to $\cdot$ are **units**. A **ring homomorphism** from R to S is a map $\varphi : R \to S$ such that $\varphi(1_R) = 1_S$, $\varphi(r + s) = \varphi(r) + \varphi(s)$, and $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$. We say R and S are isomorphic if there is a bijective ring homomorphism between them.

Definition 2.4. A **field** is a commutative ring K such that every nonzero element of K is a unit.

Example 2.5. The set $\mathbb{Z}$ of integers with the operations of addition and multiplication forms a ring. Addition is an associative and commutative operation, so $\mathbb{Z}$ is an abelian group under addition. 0 acts as the identity element with respect to addition, and for $n \in \mathbb{Z}$ we have $n+(-n) = (-n)+n = 0$. Multiplication of integers is also associative, commutative, and distributes over addition. The integers with addition and multiplication thus forms a commutative ring. The multiplicative identity in $\mathbb{Z}$ is 1. The only units in $\mathbb{Z}$ are -1 and 1, as no other integers have multiplicative inverses in $\mathbb{Z}$. $\diamondsuit$

Definition/Example 2.6. As a second example of a ring, consider the set of all K-linear maps from a vector space V over a field K to itself, or K-endomorphisms of V, denoted $\operatorname{End}_K V$. This set together with the operations of pointwise addition and function composition is a ring, called the **K-endomorphism ring** of V. The 0-map is the additive identity, and the identity map is the identity with respect to composition. $\diamondsuit$

Example 2.7. The complex numbers $\mathbb{C}$ with the operations of addition and multiplication forms a field. Addition is associative and commutative. We have 0 as an additive identity and $-1 \cdot z$ is the additive inverse of $z \in \mathbb{C}$. Therefore, $\mathbb{C}$ is an abelian group under addition. Multiplication of complex numbers is associative, commutative, and distributes over addition, so $\mathbb{C}$ is a commutative ring. The multiplicative identity is 1 and every nonzero element is a unit since $1/z$ is a complex number for all $z \neq 0$. As such, $\mathbb{C}$ is a field with these operations. $\diamondsuit$

Revisiting the concept of a group as a generalization of symmetry, we can use the language of group actions to make this characterization more concrete. Describing the action of a group on a set allows us to understand the way the set transforms with the symmetries described by the group.

Definition 2.8. An **action** of a group G on a set X is a homomorphism $\sigma : G \to S_X$, the group of permutations of X. For $g \in G, x \in X$ we denote $\sigma(g)(x)$ by $g \cdot x$. If G acts on X we call X a G-set. We say a group action is **transitive** if for all $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$. For $x \in X$ we define the **stabilizer** of x in G as $\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\}$. The set $\mathrm{Stab}_G(x)$ is a subgroup of G. The **orbit** of x under G is defined as the set $G \cdot x = \{g \cdot x : g \in G\}$. If X and Y are G-sets, then a **G-set homomorphism** is a map $\varphi : X \to Y$ such that $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $x \in X$. The G-sets X and Y are said to be equivalent if there exists a bijective G-set homomorphism between them.

Example 2.9. As an example of a G-set, consider the group S_n acting on the set $X_n = \{1, \ldots, n\}$. The homomorphism here is given by the identity function. This action is transitive, as for $j, k \in X_n$ we can define the permutation that sends $j \mapsto k$, $k \mapsto j$, and fixes everything else in X_n. $\diamondsuit$

Definition 2.10. For a group G and subgroup H, the **coset** of H in G corresponding to $g \in G$ is defined as $gH := \{gh : h \in H\}$. The set of all cosets of H in G is denoted G/H. The **index** of H in G, denoted $[G : H]$, is the number of cosets of H in G. The set G/H is a G-set, with the action given by $g' \cdot (gH) = g'gH$. If X is a transitive G-set, then X is equivalent as a G-set to $G/\mathrm{Stab}_G(x)$ with the coset action for any $x \in X$ [6].

Definition 2.11. If X and Y are G-sets, the sets $X \times Y$ and $Y \times X$ are also G-sets. Namely, we can take $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for $g \in G$ and $(x, y) \in X \times Y$, and similarly for $Y \times X$. We define the map $T : X \times Y \to Y \times X$ by $T(x, y) = (y, x)$. If $\mathcal{O}$ is an orbit of $X \times Y$, then $T(\mathcal{O})$ is clearly an orbit of $Y \times X$. Additionally, we define the following for $a \in X$ and $b \in Y$.

$$\mathcal{O}(a) = \{y \in Y : (a, y) \in \mathcal{O}\}, \quad T(\mathcal{O})(b) = \{x \in X : (x, b) \in \mathcal{O}\}.$$

The terminology of modules serves as a generalization of the familiar notion of a vector space over a field, wherein the field is replaced by a ring. Certain types of modules will prove to be particularly relevant for the remainder of this thesis.

Definition 2.12. For a ring A, an A**-module** consists of an abelian group V along with a map $A \times V \to V$, $(a, v) \mapsto a \cdot v$ that satisfies the following properties for all $a, b \in A$ and $v, w \in V$,

$$(a+b) \cdot v = a \cdot v + b \cdot v, \quad a \cdot (v+w) = a \cdot v + a \cdot w, \quad 1_A \cdot v = v.$$

If V is an A-module and W is a subgroup of V, then W is an A**-submodule** of V if $a \cdot w$ is in W for all $a \in A$ and $w \in W$. An A-module V is a **simple module** if V has exactly two A-submodules, namely $\{0\}$ and V. An A**-module homomorphism** from V to U is a map $\varphi : V \to U$ such that $\varphi(v+u) = \varphi(v) + \varphi(u)$ and $\varphi(a \cdot v) = a \cdot \varphi(v)$ for all $v, u \in V$ and $a \in A$. The A-modules V and U are isomorphic if there exists a bijective A-module homomorphism between them.

Example 2.13. Let V be a vector space over a field K. V is a K-module. The vector space V is also an $\mathrm{End}_K V$-module. For $f \in \mathrm{End}_K V$ and $v \in V$, $f \cdot v$ is given by $f(v)$. Along with these operations, V fits the definition of an $\mathrm{End}_K V$-module because each endomorphism is a linear map. $\diamondsuit$

Definition/Example 2.14. Given a set X, we construct an A-module from it as follows. For each $x \in X$, we define the function δ_x as

$$\delta_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}.$$

The set of all such δ_x forms a basis for $A^X \coloneqq \{f : X \to A\}$. Together with pointwise addition, i.e. $(f+g)(x) = f(x) + g(x)$, is an abelian group. Multiplication from A is given by $(af)(x) = a(f(x))$. Therefore A^X is an A-module, called the **free** A**-module** in X.

The observation that $\{\delta_x\}$ forms a basis for A^X leads to an alternative, and often more practical, realization of the free A-module in X. Rather than functions from X to A, we consider formal linear combinations of the form $\sum ax$, with $a \in A$. Addition and scalar multiplication are given as follows.

$$\sum a_x x + \sum b_x x = \sum (a_x + b_x) x, \quad b \sum a_x x = \sum b a_x x$$

This echoes the pointwise addition and scalar multiplication described previously. Intuitively, we can recover the function $f : X \to A$ by simply replacing x with δ_x. These constructions are isomorphic as A-modules. $\diamondsuit$

Definition 2.15. A $\mathbb{C}$**-algebra** is a $\mathbb{C}$-vector space A together with a $\mathbb{C}$-bilinear product. In other words, A has a binary operation $\cdot$ that distributes from the left and right over addition and satisfies $ax \cdot by = (ab)(x \cdot y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$. If this product is also commutative, we say that A is a **commutative** $\mathbb{C}$**-algebra**. An alternative, but equivalent, definition of a $\mathbb{C}$-algebra is a ring A together with a ring homomorphism $\lambda : \mathbb{C} \to Z(A)$ that satisfies $\lambda(1) = 1_A$. Here $Z(A)$ denotes the **center** of A, consisting of all elements of A that commute with each a in A.

Definition/Example 2.16. As an example of a $\mathbb{C}$-algebra, we can consider the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries. It is a $\mathbb{C}$-algebra with the homomorphism $\lambda : \mathbb{C} \to$

$Z(M_n(\mathbb{C}))$ given by $\lambda(z) = z \cdot I_n$, where I_n is the $n \times n$ identity matrix and $\cdot$ is the usual scalar multiplication for matrices.

Another important example of a $\mathbb{C}$-algebra is a **group algebra**. Let G be a finite group. We define the free $\mathbb{C}$-module $\mathbb{C}^G$ as in Definition/Example 2.14, with multiplication given by

$$(f_1 f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g).$$

It is straightforward to show that the operations defined above give the set of functions from G to $\mathbb{C}$ the structure of a $\mathbb{C}$-algebra.

Also as in Definition/Example 2.14, we can reinterpret the group algebra as formal sums over the group elements. Multiplication is then given by

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g b_h (gh).$$

Lastly, we define the ring homomorphism $\lambda : \mathbb{C} \to Z(\mathbb{C}G)$ by $\lambda(z) = ze$, where e is the identity element of G. With this structure, the set of sums $\sum a_g g$ is called the group algebra, denoted $\mathbb{C}G$. $\diamondsuit$

The definition that serves as the starting point for the remainder of the work here is that of a group representation. First defined by Frobenius in the late nineteenth century [3], representations serve as a powerful lens through which we can translate problems in group theory to questions in linear algebra.

Definition 2.17. A **representation** of a group G over $\mathbb{C}$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of invertible linear maps from a $\mathbb{C}$-vector space V to itself with the operation of function composition. The vector space V is then a $\mathbb{C}G$-module. If V is an n-dimensional vector space, we say ρ is a **degree** n representation. A subspace W of V such that $\rho(g)w \in W$ for all $w \in W$ and $g \in G$ is called a G**-invariant subspace**, or a $\mathbb{C}G$-submodule of V. If ρ has no nontrivial G-invariant subspaces it is called an **irreducible** representation. Any representation over $\mathbb{C}$ is a direct sum of irreducible representations [6, p. 56].

Definition 2.18. A degree n **matrix representation** of G is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices with complex entries. Now, if V is an n-dimensional $\mathbb{C}$-vector space with a basis $B = (v_1, \ldots, v_n)$, we have a group isomorphism $\mathrm{GL}(V) \to \mathrm{GL}_n(\mathbb{C})$ given by $\varphi \mapsto [\varphi]_B$, the matrix of φ with respect to B. This isomorphism gives us a way to associate a matrix representation to any representation ρ.

Example 2.19. A representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C})$ is uniquely determined by $\rho(1)$ since ρ is a group homomorphism and $\mathbb{Z}/n\mathbb{Z}$ is generated by 1. As $n1 = 0$, $\rho(1)$ must be an n^{th} root of unity. All degree 1 representations of $\mathbb{Z}/n\mathbb{Z}$ are given by $\rho_m(1) = e^{\frac{2\pi i m}{n}}$, where $1 \leq m \leq n$. These representations are also irreducible, since $\mathbb{C}$ has no nontrivial subspaces. $\diamondsuit$

Definition/Example 2.20. We can define a representation of an algebra in a similar fashion to a representation of a group. That is, a representation of a $\mathbb{C}$-algebra A on a $\mathbb{C}$-vector space V is an algebra homomorphism $\rho : A \to \operatorname{End}_{\mathbb{C}} V$. This amounts to V being an A-module.

Since A is a $\mathbb{C}$-vector space, we define a representation ζ of A on itself by

$$\zeta(a)(b) = a \cdot b$$

for $a, b \in A$. This representation is called the **regular representation** of A. $\diamondsuit$

Definition 2.21. With $\mathbb{C}G$ defined as in Definition/Example 2.16, we can establish an equivalence between representations of a finite group G and $\mathbb{C}G$-modules. If V is a $\mathbb{C}G$-module, then we define a representation $\rho_V : G \to \mathrm{GL}(V)$ by $\rho_V(g)(v) = g \cdot v$.

Closely related to the idea of a representation is that of a character. These will show up again when we address eigenvalue tables in section 4.

Definition 2.22. Given a representation ρ of a finite group G, we define the **character** of ρ as the function $\chi_\rho : G \to \mathbb{C}$ given by

$$\chi_\rho(g) = \mathrm{Trace}(\rho(g))$$

Similarly for a representation $\pi : A \to \operatorname{End}_{\mathbb{C}} V$ of a $\mathbb{C}$-algebra A, the function

$$\chi_V = \chi_\pi : A \to \mathbb{C}, \quad a \mapsto \mathrm{Trace}(\pi(a))$$

is the character of π or of V. The character of a representation is equal to the sum of the characters of the irreducible representations contained in it.

Next, we discuss an application of groups that allows us to construct $\mathbb{C}G$-modules and thus representations.

Definition 2.23. Given a G-set X, the free $\mathbb{C}$-module with X as its basis is a $\mathbb{C}G$-module if

$$\sum_{g \in G} a_g g \cdot \sum_{x \in X} b_x x = \sum_{g \in G} \sum_{x \in X} a_g b_x (g \cdot x), \quad a_g, b_x \in \mathbb{C}$$

$\mathbb{C}X$ is the **$\mathbb{C}G$-permutation module** corresponding to X. The representation $\rho_X : G \to \mathrm{GL}(\mathbb{C}X)$ associated to $\mathbb{C}X$ is called the **permutation representation** of G on X.

Example 2.24. Recall that $X_n = \{1, \ldots, n\}$ is an S_n-set. We have $\mathbb{C}X_n$ as a $\mathbb{C}S_n$-module. The corresponding permutation representation $\rho_{X_n} : S_n \to \mathrm{GL}(\mathbb{C}X_n)$ is sometimes referred to as the **natural permutation representation** of S_n. $\diamondsuit$

3. Centralizer Algebras

Given a representation $\rho : G \to \mathrm{GL}(V)$, we consider the $\mathbb{C}$-endomorphisms $T : V \to V$ that commute with the representation, i.e. $\rho(g)T = T\rho(g)$ for all $g \in G$. We can extend this idea to the corresponding permutation module. In doing so, we are able to obtain information about the irreducible representations of G.

Definition 3.1. For G-sets X and Y, we call a $\mathbb{C}$-linear map $\varphi : \mathbb{C}X \to \mathbb{C}Y$ that commutes with the action of G on each set, i.e. $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $x \in X$, a **$\mathbb{C}G$-homomorphism of permutation modules**. We denote the set of all such $\mathbb{C}G$-module homomorphisms $\mathbb{C}X \to \mathbb{C}Y$ as $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y)$. The **centralizer algebra** of X is the algebra of $\mathbb{C}G$-module homomorphisms from $\mathbb{C}X$ to itself, which we denote as $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}X)$.

Since any $\mathbb{C}G$-module homomorphism is a linear map, it has a matrix ${}_Y[\varphi]_X$ with respect to the bases X of $\mathbb{C}X$ and Y of $\mathbb{C}Y$. The entries of ${}_Y[\varphi]_X$ must be constant on the orbits of G on $Y \times X$. This leads to the following lemma regarding a basis for $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y)$. This lemma and its proof are adapted from Lux and Pahlings [6, p .28].

Lemma 3.2. *If X and Y are G-sets then*

$$\begin{aligned} Hom_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y) = \{\varphi \in Hom_{\mathbb{C}}(\mathbb{C}X, \mathbb{C}Y) : {}_Y[\varphi]_X = [a_{yx}] \\ \text{with } a_{yx} = a_{g \cdot y g \cdot x} \text{ for all } g \in G, x \in X, y \in Y\} \end{aligned}$$

For a G-orbit $\mathcal{O}$ on $Y \times X$ let $\theta_{\mathcal{O}} \in Hom_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y)$ be given by $\theta_{\mathcal{O}}(x) := \sum_{y \in T(\mathcal{O})(x)} y$. The matrix of $\theta_{\mathcal{O}}$ with respect to the bases X and Y is then ${}_Y[\varphi]_X = [a_{yx}]$ with

$$a_{yx} = \begin{cases} 0 & \text{if } (y, x) \notin \mathcal{O}, \\ 1 & \text{if } (y, x) \in \mathcal{O} \end{cases}.$$

Then $\{\theta_{\mathcal{O}} : \mathcal{O}$ a G-orbit on $Y \times X\}$ is a $\mathbb{C}$-basis for $Hom_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y)$.

Proof. Suppose $\varphi : \mathbb{C}X \to \mathbb{C}Y$ is a $\mathbb{C}G$-module homomorphism with $|X| = n$ and $|Y| = m$. Let ρ_X and ρ_Y be the permutation representations associated to $\mathbb{C}X$ and $\mathbb{C}Y$, respectively. The matrix ${}_Y[\varphi]_X = [a_{yx}] \in \mathbb{C}^{m \times n}$ of φ with respect to the bases X and Y must satisfy the following condition for all $g \in G$.

$$[\rho_Y(g)]_Y \cdot [a_{yx}] = [a_{yx}] \cdot [\rho_X(g)]_X.$$

This condition is equivalent to $a_{g^{-1}y,x} = a_{y,gx}$ for all $g \in G$, $x \in X$, and $y \in Y$, so the entries of ${}_Y[\varphi]_X$ must be constant on the orbits of G on $Y \times X$. Hence the matrices $\theta_{\mathcal{O}}$ span $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}X, \mathbb{C}Y)$. The set $\{\theta_{\mathcal{O}}\}$ is linearly independent since distinct G-orbits on $Y \times X$ are disjoint. ∎

Definition 3.3. We will be looking at the special case of $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$, i.e. with the G-set being cosets with the coset action defined in Definition 2.10, in which case the basis $\{\theta_{\mathcal{O}}\}$ constructed as above is referred to as the **Schur basis**. When $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$ is a commutative algebra, then we call the pair (G, H) a **Gelfand pair**.

Example 3.4. As an example, consider the coset action of $G = S_8 = \langle(1, 2, 3, 4, 5, 6, 7, 8), (1, 2)\rangle$ on the cosets of $H = \langle(1, 2, 3, 4, 5), (1, 2), (6, 7, 8), (6, 7)\rangle \cong S_5 \times S_3$ in G. We have 56 cosets, and so the elements of the Schur basis will be 56×56 matrices. Note that most of the computations for this example have been handled in GAP. See Appendix A for more details on the functions at use here.

```
gap> G := SymmetricGroup(8);;
gap> H := DirectProduct(SymmetricGroup(5),SymmetricGroup(3));;
gap> asc := AscendingChain(gp,subgp);;
gap> gens := GeneratorsOfGroup(gp);;
gap> res := imprim(asc, gens);;
gap> permgp := Group(res[1]);;
```

What results from the above input is `permgp`, which is the image of G under the homomorphism $G \to S_{G/H} \cong S_{56}$ that gives the action of G on G/H.

After defining the groups G and H, the following input reproduces the other four steps shown above and uses them to compute the Schur basis. While this could be accomplished by explicitly computing each orbit of $G/H \times G/H$ under G and obtaining the matrices that way, it is possible to determine them much more efficiently, which we describe.

To obtain an element of the Schur basis, the function first obtains the first row and then uses it to construct additional rows by using the permutations contained in `permgp`. Since the first row of the Schur basis consists of orbits of pairs that begin with 1, we can determine which entries should be 0 and which should be 1 for each element of the Schur basis by taking the orbit of each element of $\{1, \ldots, 56\}$ under the stabilizer of 1. Each subsequent row j can be obtained from this one by simply finding a permutation in `permgp` that maps 1 to j and applying it to the orbits under the stabilizer. This is done using `RepresentativeAction(permgp,1,j)`.

```
gap> schurbs := SchurBasisFromOrbits(G,H);;
```

This returns `schurbs`, a list containing the Schur basis for $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$. There are four elements of the Schur basis, as there are four orbits of $G/H \times G/H$ under G. These are 56×56 matrices.

Lastly, we can check to see if $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$ is commutative by simply computing $\theta_i\theta_j - \theta_j\theta_i$ for $1 \leq i, j \leq 4$. We see that (G, H) is a Gelfand pair, a fact which will be useful later on. The Schur basis will show up again when we continue in Example 3.7 on page 10. $\diamondsuit$

We are also able to find a representation of the centralizer algebra of a smaller degree. We can use the regular representation as in Definition/Example 2.20, which has the same degree as the number of orbits of the G-set. To do so, we consider how the products of the elements of the Schur basis decompose into linear combinations of basis elements. This will allow us to define the intersection numbers (also called structure constants), which is an important step in the direction of obtaining that more practical representation. For instance, moving from the Schur basis of Example 3.4 to the corresponding intersection matrices allows us to translate a problem with 56×56 matrices to 4×4 matrices.

The following theorem and its proof are adapted from Lux and Pahlings [6, p. 30].

Theorem 3.5. *For a finite G-set X, let $\mathcal{O}_i$ with $(1 \leq i \leq r)$ be the orbits of G on $X \times X$ and $\theta_i := \theta_{\mathcal{O}_i}$ be the Schur basis elements as in Lemma 3.2. Denote $\mathrm{End}_{\mathbb{C}G}\,\mathbb{C}X$ by E. Then,*

(a)

$$\theta_i\theta_j = \sum_{k=1}^{r} a_{ijk}\theta_k;\ \ a_{ijk} = |\mathcal{O}_i(x) \cap T(\mathcal{O}_j)(y)| \textit{ for } (x,y) \in \mathcal{O}_k$$

The values of a_{ijk}, known as the ***intersection numbers*** *of X, are independent of the choice of $(x,y) \in \mathcal{O}_k$.*

(b) If X is a transitive G-set, define $\theta_{T(i)} := \theta_{T(\mathcal{O}_i)}$. Then,

$$a_{iT(j)1} = |\mathcal{O}_i(x)|\delta_{ij}$$

for any $x \in X$. Here, δ_{ij} is defined as 0 if $i \neq j$ and 1 if $i = j$. For $\mathcal{O}_1 = \{(x,x) : x \in X\}$, $\theta_1 = 1_E$.

Proof. (a) Let $n = |X|$. Let $[a^k_{xy}]_{x,y\in X}$ be the $n \times n$ matrix of θ_k with respect to the basis X of $\mathbb{C}X$. Applying Lemma 3.2, we know that

$$a^k_{xy} = \begin{cases} 0 & \text{if } (x,y) \notin \mathcal{O}_k, \\ 1 & \text{if } (x,y) \in \mathcal{O}_k \end{cases}.$$

The entry (x,y) of the matrix of ${}_X[\theta_i\theta_j]_X$ with respect to X is as follows.

$$\begin{aligned} \sum_{z\in X} a^i_{xz}a^j_{zy} &= \sum_{z\in\mathcal{O}_i(x)} a^j_{zy} \\ &= \sum_{z\in\mathcal{O}_i(x)\cap T(\mathcal{O}_j)(y)} 1_{\mathbb{C}} \\ &= |\mathcal{O}_i(x) \cap T(\mathcal{O}_j)(y)| \end{aligned}$$

This value does not depend on which $(x,y) \in \mathcal{O}_k$ is selected.

(b) This follows from the proof of part (a), as $T(\mathcal{O}_{T(j)})(y) = \mathcal{O}_j(y)$. ■

Now, the main use of this theorem for our purposes is obtaining the intersection matrices.

Definition 3.6. Given a finite G-set X with orbits $\mathcal{O}_i$, $1 \leq i \leq r$, on $X \times X$ and using notation as above, we define the **intersection matrix** A_i as $[a_{ijk}]_{1\leq j,k\leq r}$.

The map $\operatorname{End}_{\mathbb{C}G}\mathbb{C}X \to \mathbb{C}^{r\times r}$ given by $\theta_i \mapsto A_i$ gives us the matrix representation corresponding to the regular representation ζ of $\operatorname{End}_{\mathbb{C}G}\mathbb{C}X$ described in Definition/Example 2.20. This follows from the intersection matrices consisting of the coefficients of the Schur basis elements in their products. It is also worth mentioning that we can compute the intersection matrices without ever having to look at the Schur basis as matrices, since we have a convenient combinatorial description of their entries.

Example 3.7. We continue with Example 3.4, where we have $G = S_8$ acting on the cosets of $H = S_5 \times S_3$ in G. Our next step is to obtain the intersection matrices for G/H using GAP and hence the regular representation of its centralizer algebra. Variables here are defined as before, most importantly with `permgp` the permutation group for the cosets being acted on by G.

```
gap> orbs := Orbits(permgp, Tuples([1..Index(G,H)],2), OnPairs);;
gap> intmats := IntersectionMatrices(orb, Index(G,H));;
```

So, now we have the list `intmats` of intersection matrices. We can look at these by using `Display(intmats[j])` for $j = 1, \ldots, 4$, since 4 is the number of orbits of G on $G/H \times G/H$. Doing so shows us the following four intersection matrices.

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 15 & 6 & 4 & 0 \\ 0 & 8 & 8 & 9 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 8 & 8 & 9 \\ 30 & 16 & 15 & 18 \\ 0 & 6 & 6 & 3 \end{pmatrix}; \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 6 & 6 & 3 \\ 10 & 4 & 1 & 0 \end{pmatrix}$$

So, we have determined the intersection matrices for G/H. Since the centralizer algebra itself is commutative and the regular matrix representation is given by $\theta_i \mapsto A_i$, we know that the intersection matrices commute as well. In Example 4.2 on page 11 we will make use of these matrices to compute the eigenvalue table (also called the character table) for $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$. $\diamondsuit$

4. Eigenvalue Tables of Centralizer Algebras

When G acts on G/H, where (G, H) is a Gelfand pair, we can construct the character table for the centralizer algebra that can be used to extract information about the irreducible representations of G. When (G, H) is a Gelfand pair, the centralizer algebra of (G, H) is commutative. The intersection matrices also commute in this case.

Theorem 4.1. *Let A and B be diagonalizable operators from $V \to V$. If A and B commute, they are simultaneously diagonalizable, i.e. there is a basis of V consisting of common eigenvectors of A and B.*

Definition/Example 4.2. Once again, we return to the example of $G = S_8$ acting on the cosets of $H = S_5 \times S_3$ in G. In Example 3.7, we computed the intersection matrices for this G-set and established that they commute. It is straightforward to check with GAP that the intersection matrices are all diagonalizable. By Theorem 4.1, this tells us that there is a basis of $\mathbb{C}^4$ consisting of common eigenvectors of the intersection matrices. The following vectors are one such eigenbasis.

$$[1, 15, 30, 10]^T, \; [1, 7, -2, -6]^T, \; [1, 1, -5, 3]^T, \; [1, -3, 3, -1]^T$$

These vectors were obtained by using the `Eigenspaces` function in GAP for each of the intersection matrices.

Each eigenspace corresponds to a subspace of $\mathbb{C}^4$ that is $E := \mathrm{End}_{\mathbb{C}G}(\mathbb{C}(G/H))$-invariant, since they are eigenspaces for all of the intersection matrices. Therefore, the centralizer algebra E has four degree one irreducible representations, denoted ζ_i. With this information in hand, we can construct the eigenvalue table of E.

The **eigenvalue table** of E has rows labeled by the irreducible representations corresponding to the eigenspaces ζ_i and columns labeled by the elements of the Schur basis of E, θ_i. The number at position (ζ_i, θ_j) in the table is the eigenvalue is the eigenvalue of the vector corresponding to ζ_i for the Schur basis element θ_j.

	θ_1	θ_2	θ_3	θ_4
ζ_1	1	15	30	10
ζ_2	1	7	-2	-6
ζ_3	1	1	-5	-3
ζ_4	1	-3	3	-1

When we revisit this example on page 13, we will make use of this eigenvalue table to obtain information about the irreducible representations of S_8. ♢

Definition 4.3. If A is a $\mathbb{C}$-algebra, we say $e \in A$ is an **idempotent** if $e^2 = e$. The idempotents $e_1, e_2 \in A$ are said to be orthogonal if $e_1 e_2 = e_2 e_1 = 0$. A **block idempotent** of A is an idempotent in the center of A that cannot be written as the sum of two orthogonal idempotents in the center of A.

Having established that eigenvalue tables can be constructed in the case where (G, H) is a Gelfand pair, we can now look at some of their applications. One such application is determining degrees of the irreducible representations of G in the permutation representation on G/H. The following theorem details how this can be accomplished.

Theorem 4.4. *Let Ω be a transitive G-set. Then*

$$\mathbb{C}\Omega \cong_{\mathbb{C}G} \bigoplus_{i=1}^{m} \underbrace{L_i \oplus \cdots \oplus L_i}_{n_i}, \quad L_i \not\cong_{\mathbb{C}G} L_j \text{ for } i \neq j$$

with simple $\mathbb{C}G$-modules L_i. We have $\operatorname{End}_{\mathbb{C}G} L_i \cong \mathbb{C}$. *Denote* $\dim_{\mathbb{C}} L_i$ *by z_i. Then*

$$E := \operatorname{End}_{\mathbb{C}G} \mathbb{C}\Omega = \bigoplus_{i=1}^{m} E\epsilon_i, \quad E\epsilon_i \cong D_i^{n_i \times n_i}$$

with block idempotents ϵ_i. Let $(\theta_1 = 1, \ldots, \theta_m)$ be the Schur basis of E and let ζ be the character of the E-module $\mathbb{C}\Omega$. Assume that ζ_i is the character of the simple E-module in $E\epsilon_i$. Then $\zeta = \sum_{i=1}^{m} z_i \zeta_i$ and

$$\epsilon_i = z_i \sum_{j=1}^{m} \frac{\zeta_i(\theta_{T(j)})}{|\mathcal{O}_j|} \theta_j \tag{4.1}$$

$$\sum_{k=1}^{m} \frac{1}{|\mathcal{O}_k|} \zeta_i(\theta_k) \zeta_j(\theta_{T(k)}) = \frac{\zeta_i(1)}{z_i} \delta_{ij} \tag{4.2}$$

Proof. A proof of this theorem in its entirety can be found in Lux and Pahlings [6, p. 90]. Equations 4.1 and 4.2 are of particular interest to us, and so we will include their proofs here.

Corollary 1.2.22 on page 31 of Lux and Pahlings shows that $\zeta(\theta_i\theta_{T(j)}) = \delta_{ij}|\mathcal{O}_i|$. If we write the idempotent ϵ_i as $\sum_{j=1}^m \alpha_j\theta_j$ with $\alpha_j \in \mathbb{C}$, we have

$$\begin{aligned}\alpha_j|\mathcal{O}_j| &= \zeta(\epsilon_i\theta_{T(j)})\\ &= \sum_{k=1}^m z_k\zeta_k(\epsilon_i\theta_{T(j)})\\ &= z_i\zeta_i(\theta_{T(j)})\end{aligned}$$

Therefore, we have the following equation for each coefficient α_j.

$$\alpha_j = \frac{z_i\zeta_i(\theta_{T(j)})}{|\mathcal{O}_j|}$$

Equation 4.1 follows.

For the proof of Equation 4.2, we first make use of the fact that the ϵ_i are orthogonal idempotents. This means that $\delta_{ij}\epsilon_i = \epsilon_i\epsilon_j$. We get the following equation from this.

$$\begin{aligned}\delta_{ij}\epsilon_i &= z_iz_j\sum_{k=1}^m\sum_{l=1}^m \frac{\zeta_i(\theta_{T(k)})\zeta_j(\theta_{T(l)})}{|\mathcal{O}_k||\mathcal{O}_l|}\theta_k\theta_l\\ &= z_iz_j\sum_{k=1}^m\sum_{l=1}^m\sum_{u=1}^m \frac{\zeta_i(\theta_{T(k)})\zeta_j(\theta_{T(l)})}{|\mathcal{O}_k||\mathcal{O}_l|}a_{klu}\theta_u\end{aligned}$$

We can compare the coefficients of θ_1, the identity element of E and apply part (b) of Theorem 3.5 to obtain another equation.

$$z_i\frac{\zeta_i(1)}{|\mathcal{O}_1|}\delta_{ij} = z_iz_j\sum_{k=1} \frac{\zeta_i(\theta_{T(k)})\zeta_j(\theta_k)}{|\mathcal{O}_k|^2}\frac{|\mathcal{O}_k|}{|\mathcal{O}_1|}$$

After rearranging and cancelling, we are left with Equation 4.2. ■

Example 4.5. We continue with the eigenvalue table computed in Definition/Example 4.2. We will make use of Theorem 4.4 to determine degrees of three nontrivial irreducible representations of the symmetric group S_8. These degrees can be obtained by solving Equation 4.2 for the z_i.

We showed in Definition/Example 4.2 that the degrees of the irreducible representations ζ_i are all 1, and so $\zeta_i(1) = 1$ for $i = 1, \ldots, 4$. When $i = j$, we have

$$\frac{1}{z_i} = \sum_{k=1}^4 \frac{\zeta_i(\theta_k)\zeta_i(\theta_{T(k)})}{|\mathcal{O}_k|}$$

To simplify this one step further, we can use GAP to find that the orbits of $G/H \times G/H$ under G have sizes 56, 560, 840, and 1680. Since these are all distinct we know that $\mathcal{O}_k = T(\mathcal{O}_k)$ for all k. Therefore,

$$\frac{1}{z_i} = \sum_{k=1}^4 \frac{(\zeta_i(\theta_k))^2}{|\mathcal{O}_k|}$$

We can read the values of $\zeta_i(\theta_k)$ off of the eigenvalue table obtained in Definition/Example 4.2, so the degrees of the irreducible representations of S_8 in the decomposition of this permutation representation into irreducibles are 1, 7, 20, and 28, respectively. $\diamondsuit$

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Appendix A. GAP Code

This function takes a group G and subgroup H as arguments and computes the Schur basis for the centralizer algebra of the cosets G/H as defined in Lemma 3.2. The function makes use of `imprim`, which was obtained from Dr. Klaus Lux.

```
SchurBasisFromOrbits := function(gp, subgp)
    local orbs, asc, gens, res, permgp, rowsw,
        i, j, k, x, row, orbmat, orbnr, neworb,
        schurbs, idx, stab;

    # initial setup
    asc := AscendingChain(gp,subgp);
    gens := GeneratorsOfGroup(gp);
    res := imprim(asc, gens);
    permgp := Group(res[1]);
    stab := Stabilizer(permgp,1);
    idx := Index(gp,subgp);

    # compute the orbits of everything under the stabilizer
    orbs := Orbits(stab, [1..idx]);
    orbnr := Size(orbs);
    schurbs := [];

    # construct the Schur basis
    for i in [1..orbnr] do
```

```
        orbmat := [];

        # construct all the rows using row one
        for j in [1..idx] do
            row := [];
            rowsw := RepresentativeAction(permgp,1,j);
            neworb := List(orbs[i], x -> x^rowsw);

            for k in [1..idx] do
                if k in neworb then
                    Append(row,[1]);
                else
                    Append(row,[0]);
                fi;
            od;

            Append(orbmat,[row]);

        od;

        Append(schurbs,[orbmat]);

    od;

    return schurbs;

end;
```

This function continues with the idea of centralizer algebras for G/H for some group G and subgroup H. It takes as arguments the orbits of G/H under G and the index of H in G, and computes the corresponding intersection matrices as defined in theorem 3.5.

```
IntersectionMatrices := function(orb, idx)
    local a, x, y, i, j, k;

    a := [];; x := 1;; y := 1;;

    for i in [1..Length(orb)] do
        a[i] := []; # a[i] will be the i-th intersection matrix
        for j in [1..Length(orb)] do
            a[i][j] := []; # a[i][j] will be the j-th row of a[i]
            for k in [1..Length(orb)] do
                x := orb[k][1][1]; y := orb[k][1][2]; # [x,y] in orb[k]
                a[i][j][k] := Size( Intersection (
                    Filtered([1..idx] , z -> [x,z] in orb[i]),
                    Filtered([1..idx] , z -> [y,z] in orb[j]) ) );
            od;
        od;
    od;
```

```
    return a;

end;
```

Using the intersection matrices obtained with the previous script, the first function in this script produces the eigenvalue table for the centralizer algebra of G/H. The second function takes the eigenvalue table and index of H in G and obtains the character degrees as in Equation 4.2 from Theorem 4.4. Note that this script is only functional when (G, H) is a Gelfand pair, as otherwise the matrices are not simultaneously diagonalizable.

```
GelfandEigenvalueTable := function(mats)
    # mats = intersection matrices
    local i, j, eigtb, eigsp, vec, eigvec, comm, zero;

    # check to make sure that the intersection matrices originate
    # from a Gelfand pair (G,H)
    zero := NullMat(Size(mats),Size(mats));
    for i in [1..Size(mats)] do
        for j in [i..Size(mats)] do
            comm := mats[i] * mats[j] - mats[j] * mats[i];
            if comm <> zero then
                Error("(G,H) is not a Gelfand pair.");
            fi;
        od;
    od;

    # get the eigenspaces of the intersection matrices
    eigsp := List(mats, x -> Eigenspaces(Rationals, TransposedMat(x)));

    eigtb := [];

    for i in [1..Size(mats)] do
        eigtb[i] := []; # will become the i-th column
        for j in [1..Size(mats)] do
            eigvec := Representative(eigsp[2][i]);
            vec := mats[j] * eigvec;
            eigtb[i][j] := vec[1] / eigvec[1];
        od;
    od;

    return eigtb;

end;

GelfandCharacterDegrees := function(eigtb, idx)
    # gets the character degrees of the L_i per notation from thm 2.1.8
    # eigtb = eigenvalue table of a centralizer algebra w/ Gelfand
    #   situation
```

```
    # idx = the index of sg in g
    local i, j, val, sum, sizes, degs;

    degs := [];
    sizes := List(eigtb[1], x -> x * idx);

    for i in [1..Size(eigtb)] do
        val := 0;
        for j in [1..Size(eigtb[i])] do
            val := val + (eigtb[i][j])^2 / (sizes[j]);
        od;
        Append(degs, [1/val]);
    od;

    return degs;

end;
```